

Asymptotics of step-like solutions for the Camassa-Holm equation

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Abstract

We study the long-time asymptotics of solution of the Cauchy problem for the Camassa-Holm equation with a step-like initial datum.

By using the nonlinear steepest descent method and the so-called g -function approach, we show that the Camassa-Holm equation exhibits a rich structure of sharply separated regions in the x, t -half-plane with qualitatively different asymptotics, which can be described in terms of a sum of modulated finite-gap hyperelliptic or elliptic functions and a finite number of solitons.

Keywords: Camassa-Holm equation, Riemann-Hilbert problem, step-like initial data, g -function approach

1. Introduction

We consider the Cauchy problem for the Camassa – Holm (CH) equation on the line with a step-like initial datum

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad -\infty < x < \infty, \quad t > 0, \quad (1.1)$$

$$u(x, 0) = u_0(x), \text{ where} \quad (1.2)$$

$$\begin{cases} u_0(x) \rightarrow c > 0, & x \rightarrow -\infty, \\ u_0(x) \rightarrow 0, & x \rightarrow +\infty. \end{cases} \quad \omega > 0, \quad (1.3)$$

We consider real-valued solutions $u(x, t)$. Based on the Riemann-Hilbert problem formalism [35], we study here the long-time asymptotics for the solution of the initial value problem.

The Camassa-Holm equation describes the unidirectional propagation of shallow water waves over a flat bottom ([16], [17]) as well as axially symmetric waves in a hyperelastic rod ([22]). First, it was found using the method of recursion operators as a bi-Hamiltonian equation with an infinite number of conserved functionals ([26]).

Study of initial-value problems for integrable equations with step-like initial data originates from the seminal papers [29], [30], see also [31], [1], [2], [3]. However, an implementation of the rigorous Riemann-Hilbert problem scheme to step-like initial value problems was done only recently in [14], [6], and now this area is actively being developed (see [32], [33], [24], [25], [34] and references therein). The importance of such studies is that similar methods have been applied in studies of semiclassical problems [13], Painleve functions [12], random matrix models [18], asymptotics of the determinant of Fredholm operator with sine kernel [5].

We must mention that an alternative way of asymptotic analysis, which does not employ the fact of fully integrability of equation, can be made by using dispersive methods based on Duhamel's formula and Strichartz estimates (cf. [37], [39], [36], [27]).

However, for the CH equation the only asymptotic analysis was done by using Riemann-Hilbert method and nonlinear steepest descent method [7], [8], [9], [10], [11]. This analysis was restricted to the case of vanishing initial data, and the aim of the present paper is to generalize those results to the case of step-like initial data.

In the sequel we made the following assumptions on the initial datum and solution of the Cauchy problem (1.1), (1.2).

Assumption 1. We restrict ourselves to the following class of initial conditions:

$$c > 0, \omega > 0, \quad m_0(x) + \omega > 0, \quad (1.4)$$

where $m_0(x) := u_0(x) - u_{0,xx}(x)$.

Comment 1.1. In [35, Lemma 2.1] it is shown that Assumption 1 provides

$$m(x, t) + \omega > 0 \quad (1.5)$$

for all values of time $t \geq 0$, where the function

$$m(x, t) := u(x, t) - u_{xx}(x, t)$$

is the so-called “momentum” variable. Our method handles only solutions satisfying this property (1.5).

Comment 1.2. The class of initial data (1.3) is equivalent to the following class of initial data:

$$u_0(x) \rightarrow \begin{cases} c_l, & \text{as } x \rightarrow -\infty, \\ c_r, & \text{as } x \rightarrow +\infty, \end{cases} \quad \omega \in \mathbb{R},$$

where

$$\frac{c_l + \omega}{c_r + \omega} > 0, \quad \frac{m_0(x) + \omega}{c_r + \omega} > 0.$$

This can be seen by using the following change of variables, which transforms a solution of the CH equation into solution of the CH equation ([28], p.4):

$$(\omega, u(x, t)) \mapsto (\alpha\omega - \beta, v(x, t) = \alpha u(x - \beta t, \alpha t) + \beta).$$

This transformation preserves the quantity $\frac{c_l + \omega}{c_r + \omega}$ and the function $\frac{m(x, t) + \omega}{c_r + \omega}$.

Assumption 2. We assume that

$$\forall x \in \mathbb{R} : \quad c \geq m_0(x) \quad (1.6)$$

with c from (1.3).

Comment 1.3. Lemma 2.3 (cf. [35, Lemma 2.6]) below provides that in this case the corresponding Wronskian of the associated Jost solutions does not vanish at the edge of the single spectrum: $W(i\hat{c}) \neq 0$ (see Section 2 for details) and hence, the (right) transmission coefficient takes continuous boundary values at the edge of one-fold spectrum $k = i\hat{c}$.

Assumption 3. Moreover, we assume that the initial condition tends to its limits as $x \rightarrow \pm\infty$ exponentially fast,

$$\int_{\mathbb{R}} e^{C_0|x|} (|m(x, 0) - cH(-x)| + |m_x(x, 0)| + |m_{xx}(x, 0)|) dx < \infty, \quad (1.7)$$

where $C_0 > \frac{c}{4(c + \omega)}$ with c from (1.3).

Comment 1.4. This assumption provides analytic continuation of the corresponding spectral functions $a(k)$, $b(k)$ and reflection coefficient $r(k) = \frac{b(k)}{a(k)}$ into some neighborhood of the contour of Riemann-Hilbert problem $\Sigma = \mathbb{R} \cup [i\hat{c}, -i\hat{c}]$ (with \hat{c} from (2.11)).

The existence of a *weak* solution to step-like initial value problems for the Camassa-Holm equation as well as for other nonlinear systems can be established by using PDE techniques [28]. However, our method is suitable to work only with classical solutions, and hence, we have to assume the existence of a global classical solution (see Assumption 4 below). Although we must assume the existence of a *classical* solution to this problem, the advantage of our approach is that it yields a rigorous asymptotic analysis by using an extension of the nonlinear steepest-descent method [23].

Assumption 4. Suppose that there exists a global real-valued classical solution $u(x, t)$ of the CH equation (1.1), which tends rapidly to its limits as $x \rightarrow \pm\infty$, that is, for any $T \geq 0$

$$\max_{0 \leq t \leq T} \int_{-\infty}^{+\infty} (1 + |x|) \times (|m(x, t) - cH(-x)| + |m_x(x, t)| + |m_{xx}(x, t)|) dx < \infty,$$

where $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is the Heaviside function. (1.8)

Comment 1.5. This assumption establishes the existence of the corresponding Jost solutions for all values of time $t \geq 0$ (see Section 2 for details).

The existence of solutions of the Cauchy problem (1.1), (1.2) satisfying Assumptions 1-4 can be established by extensions of the methods, used for the case of vanishing initial data (see for example [20]). Since our main goal is the asymptotics, we will not consider this issue here. On the other hand, we know that the class of solutions satisfying the Assumptions 1-4 is not empty. Indeed, by standard method one can check that each function which is obtained from the RH problem in Section 2 by formulas (2.31), is a solution of the CH equation (see for example [38], [8, Theorem 5.2]).

• **Main result.**

The main result is that the $x, t \geq 0$ half-plane is divided into several domains with qualitatively different asymptotics of solution. Namely, if we look at asymptotics with accuracy up to a decaying term, then we have 3 different sectors, and if we look for asymptotics with accuracy up to $t^{-1/2}$, then we have 5 different sectors (see Figures 1, 2, 3).

This subdivision considerably depends on whether $\frac{c}{\omega} > 3$, $1 < \frac{c}{\omega} < 3$, or $0 < \frac{c}{\omega} < 1$.

Parameters $\zeta_j, j = 1, \dots, 4$ depend on the value of $\frac{c}{\omega}$ and are defined in (3.40)-(3.41).

The asymptotics in the domains marked in yellow in Figures 1–3 is described by formulas (4.76), (4.77) in Theorem 4.1, except for the domain D_{2a} in the case $\frac{c}{\omega} > 3$; in the latter case the asymptotics is described by formulas (4.78), (4.79) in Theorem 4.2. The asymptotics in the domains marked in purple is described by Theorem 4.3. Finally, the asymptotics in the domains marked in green is described by Theorem 5.1.

Briefly, asymptotics in "yellow" domains is described by modulated elliptic (genus 1) or hyperelliptic (genus 2) functions, in "purple" domains the leading asymptotic term is the constant c as in (1.3), and in "green" domains we have solitonic asymptotics on a vanishing background.

The paper is organized as follows: in Section 2 we list some facts about the Camassa-Holm equation and formulate the vector Riemann-Hilbert problem. In Section 3 we describe the corresponding g -function approach and the suitable phase functions that we use in asymptotic analysis in different regions of the $x, t \geq 0$ -half-plane. In Sections 4, 5 we give a chain of Riemann-Hilbert problem transformations, which leads to model problems explicitly solvable in terms of elliptic (genus 1) or hyperelliptic (genus 2) functions. This in

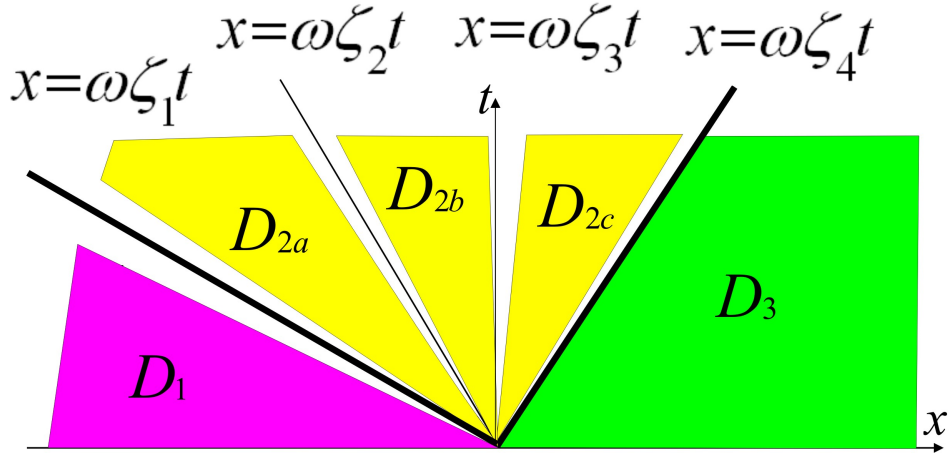


Figure 1: Case $\frac{c}{\omega} > 3$. Regions in x, t -halfplane with qualitatively different asymptotics.

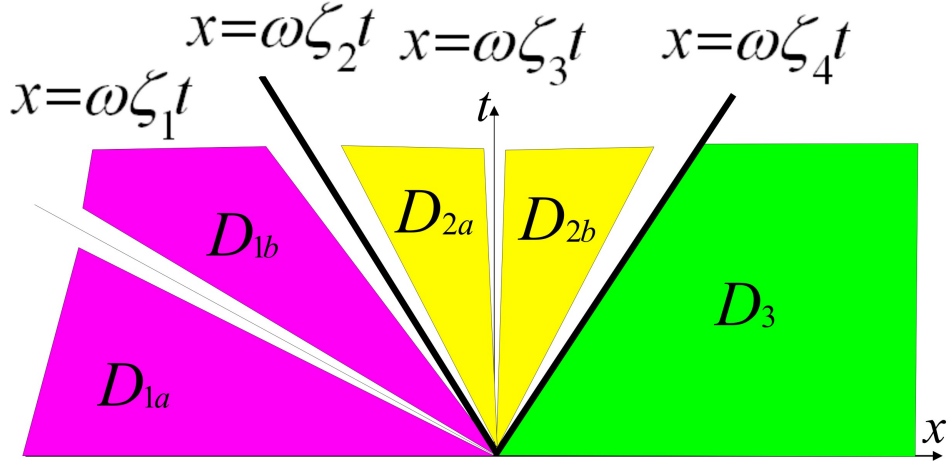


Figure 2: Case $1 < \frac{c}{\omega} < 3$. Regions in x, t -halfplane with qualitatively different asymptotics.

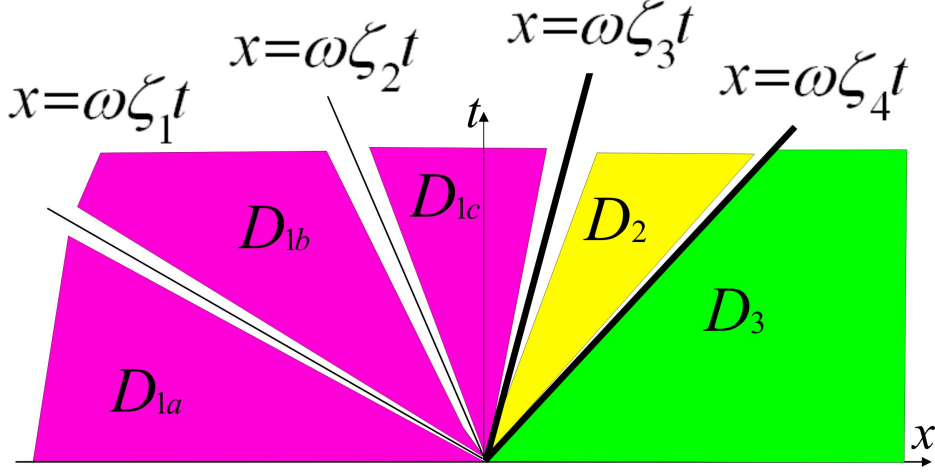


Figure 3: Case $0 < \frac{c}{\omega} < 1$. Regions in x, t -halfplane with qualitatively different asymptotics.

turn gives us explicit expressions for the asymptotics.

Acknowledgements. We thank Iryna Egorova, Dmitry Shepelsky, Robert Buckingham and Svetlana Roudenko for useful discussions.

The research has been supported by the project "Support of inter-sectoral mobility and quality enhancement of research teams at Czech Technical University in Prague", CZ.1.07/2.3.00/30.0034, sponsored by European Social Fund in the Czech Republic."

2. Preliminaries

2.1. Lax pair, Jost solutions, spectral functions

We begin by recalling some important results for the Camassa – Holm equation [7], [10], [20], [21], [35].

The starting point for our considerations is the Lax pair representation: the CH equation is the compatibility condition of two linear equations

$$-\varphi_{xx}'' + \frac{1}{4}\varphi = \lambda \frac{m + \omega}{\omega} \varphi, \quad (2.9a)$$

$$\varphi_t = -\left(\frac{\omega}{2\lambda} + u\right)\varphi_x' + \frac{u_x}{2}\varphi, \quad (2.9b)$$

$$\text{where} \quad \lambda =: k^2 + \frac{1}{4} =: z^2 + \frac{\omega}{4(c+\omega)} \quad (2.10)$$

are the spectral parameters. This means that $\varphi_{xxt} = \varphi_{txx}$ iff u satisfies the CH equation.

The x - equation is closely related to the spectral problem for the Schrodinger operator with a step-like potential, which has been studied in *Buslaev, Fomin* [15], *Cohen, Kappeler* [19].

$$\text{Denote} \quad \hat{c} = \frac{1}{2} \sqrt{\frac{c}{c+\omega}} \in \left(0, \frac{1}{2}\right). \quad (2.11)$$

The next Lemma 2.1 (cf. [35, Lemma 2.1]) distinguishes an important class of solutions of the Camassa-Holm equation (see Assumption 1 in Section 1).

Lemma 2.1. (cf. [20], [35, Lemma 2.1]).

Suppose there exists a global classical solution to the problem (1.1), (1.2). Given Assumption 1, we have

$$m(x, t) + \omega > 0 \quad (2.12)$$

for all values of time $t \geq 0$.

This lemma justifies consideration of solutions of the Camassa-Holm equation (1.1) satisfying $m(x, t) + \omega > 0$ for all x and t , so from now on, we consider only solutions satisfying (2.12). Then (1.1) can be written in a form of a local conservation law

$$\left(\sqrt{\frac{m+\omega}{\omega}} \right)_t = - \left(u \sqrt{\frac{m+\omega}{\omega}} \right)_x.$$

There are indeed infinite number of conservation laws, in the sequel we need two of them:

$$\begin{aligned} H_{-1} = & x \left(\sqrt{\frac{c+\omega}{\omega}} - 1 \right) - c \sqrt{\frac{c+\omega}{\omega}} t + \int_{-\infty}^x \left(\sqrt{\frac{m(\xi, t) + \omega}{\omega}} - \sqrt{\frac{c+\omega}{\omega}} \right) d\xi + \\ & + \int_x^{+\infty} \left(\sqrt{\frac{m(\xi, t) + \omega}{\omega}} - 1 \right) d\xi \quad (2.13) \end{aligned}$$

$$H_0 = \int_{-\infty}^x (c - m(r, t)) dr - \int_x^{+\infty} m(r, t) dr - c(x+1) + \frac{c(3c+4\omega)t}{2}. \quad (2.14)$$

An important role is played by the quantity

$$\begin{aligned} y &:= x - \int_x^{+\infty} \left(\sqrt{\frac{m+\omega}{\omega}} - 1 \right) dr \\ &= -H_{-1} + \sqrt{\frac{c+\omega}{\omega}} \left[x - ct + \int_{-\infty}^x \left(\sqrt{\frac{m+\omega}{c+\omega}} - 1 \right) dr \right]. \end{aligned} \quad (2.15)$$

For further use let us notice that $y \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$ and vice versa.

Lemma 2.2. (cf. [35, Lemma 2.2, 2.3]) *Assumption 4 ensures that there exist two Jost solutions $\varphi_l(x, t; k)$, $\varphi_r(x, t; k)$ that solve both the x - and t -equations (2.9a), (2.9b), and satisfy*

$$\lim_{x \rightarrow -\infty} \sqrt[4]{\frac{c+\omega}{\omega}} \exp \left\{ iz \sqrt{\frac{c+\omega}{\omega}} \left(x - \left(c + \frac{\omega}{2\lambda} \right) t \right) \right\} \varphi_l(x, t; k) = 1, \quad (2.16)$$

$$\lim_{x \rightarrow +\infty} e^{-ikx + \frac{i\omega kt}{2\lambda}} \varphi_r(x, t; k) = 1. \quad (2.17)$$

The function $\exp \left\{ \frac{-izt\sqrt{(c+\omega)\omega}}{2\lambda} \right\} \varphi_l(x, t; k)$ is analytic in

$$D := \{k : \text{Im} k > 0\} \setminus [0, i\hat{c}] \quad (2.18)$$

and continuous in (subscripts “ \pm ” denote the right/left side of the bank)

$$\overline{D} := (\{k : \text{Im} k \geq 0\} \setminus [0, i\hat{c}]) \cup [0, i\hat{c}]_- \cup [0, i\hat{c}]_+; \quad (2.19)$$

The last expression means that we distinguish between the left and right banks of the segment $[0, i\hat{c}]$. Here,

- $\varphi_l(x, t; \cdot)$ is discontinuous across $[0, i\hat{c}]$,
- the function $\exp \left\{ \frac{ikt\omega}{2\lambda} \right\} \varphi_r(x, t; k)$ is analytic for $\text{Im} k > 0$ and continuous for $\text{Im} k \geq 0$. As $k \rightarrow \infty$, $\text{Im} k \geq 0$, we have

$$\begin{aligned} \bullet \quad \varphi_l(x, t; k) &= \sqrt[4]{\frac{\omega}{m+\omega}} \cdot \\ &\cdot \exp \left\{ -iz \sqrt{\frac{c+\omega}{\omega}} \left(x + \int_{-\infty}^x \left(\sqrt{\frac{m+\omega}{c+\omega}} - 1 \right) d\tilde{x} - \left(c + \frac{\omega}{2\lambda} \right) t \right) \right\} \cdot \\ &\cdot \left(1 - \frac{1}{2ik} \int_{-\infty}^y \left(v(\tilde{y}, t) + \frac{c}{4(c+\omega)} \right) d\tilde{y} + O(k^{-2}) \right), \end{aligned} \quad (2.20)$$

$$\begin{aligned}
\bullet \quad \varphi_{\tau}(x, t; k) &= \sqrt[4]{\frac{\omega}{m + \omega}} \exp \left\{ ik \left(x - \int_x^{+\infty} \left(\sqrt{\frac{m + \omega}{\omega}} - 1 \right) d\tilde{x} \right) - \frac{i\omega kt}{2\lambda} \right\} \\
&\cdot \left(1 - \frac{1}{2ik} \int_y^{+\infty} v(\tilde{y}, t) d\tilde{y} + O(k^{-2}) \right). \tag{2.21}
\end{aligned}$$

Moreover, the following relations are satisfied for $k \in (i\hat{c}, 0)$:

$$\begin{aligned}
\varphi_{l,\tau}(x, t; k) &= \overline{\varphi_{l,\tau}(x, t; -\bar{k})}, \quad \varphi_l(x, t; k \pm 0) = \overline{\varphi_l(x, t; \bar{k} \mp 0)}, \\
\varphi_{\tau}(x, t; k) &= \overline{\varphi_{\tau}(x, t; \bar{k})}, \quad \varphi_l(x, t; k \pm 0) = \overline{\varphi_l(x, t; \bar{k} \pm 0)}.
\end{aligned}$$

Let us notice, that D , \overline{D} , and

$$\partial\overline{D} := \mathbb{R} \cup [0, i\hat{c}]_- \cup [0, i\hat{c}]_+ \tag{2.22}$$

can be characterized in terms of $z(k)$ (2.10) as the domains where $\text{Im}z(k) > 0$, $\text{Im}z(k) \geq 0$, and both $\text{Im}z(k) = 0$ and $\text{Im}k \geq 0$, respectively.

The Jost solutions determine spectral functions via the scattering relation

$$\varphi_l(x, t; k) = a(k) \overline{\varphi_{\tau}(x, t; \bar{k})} + b(k) \varphi_{\tau}(x, t; k), \quad k \in \mathbb{R} \setminus \{k = 0\}, \tag{2.23a}$$

which determines the (right) transmission coefficient $a^{-1}(k)$, and the (right) reflection coefficient $r(k) := \frac{b(k)}{a(k)}$.

In the following preliminary lemma we use the notations (2.18), (2.19), (2.22) for the domains D , \overline{D} , and $\partial\overline{D}$.

Lemma 2.3. (cf. [35, Lemmas 2.4-2.7])

- The (right) transmission coefficient $a^{-1}(k)$ is meromorphic in D ; has a finite number of poles $i\kappa_1, \dots, i\kappa_N$, which lie in the interval $\hat{c} < \kappa_N < \dots < \kappa_1 < \frac{1}{2}$; function $a^{-1}(k)$ is continuous up to the boundary of its domain of analyticity with (in general) the exception of the edge point $k = i\hat{c}$. Moreover, $a_{\pm}^{-1}(0) = 0$.
- Further, under Assumption 2 ($\forall x \in \mathbb{R} : c \geq m_0(x)$), the (right) transmission coefficient $a^{-1}(k)$ is continuous up to the point $k = i\hat{c}$;
- Under Assumption 3, the reflection coefficient $r(k) \equiv \frac{b(k)}{a(k)}$ is meromorphic in $D_{C_0} := \{k : 0 < \text{Im}k < \sqrt{C_0}\} \setminus \{i/2\}$ with simple poles at $i\kappa_j$ (those which lie in the domain D_{C_0}), and takes continuous boundary values at $(i\hat{c}, 0]$ (recall

that C_0 is the constant that appears in (1.7), Assumption 3). Further, under Assumption 2, $r(k)$ is continuous up to the point $k = i\hat{c}$;

- Asymptotically as $k \rightarrow \infty$ we have

$$a^{-1}(k) = e^{iH^{-1}k} (1 + O(k^{-1})) , \quad r(k) \equiv \frac{b(k)}{a(k)} = O(k^{-1}). \quad (2.24)$$

- The residues of $a_+^{-1}(k)$ are given by

$$\text{Res}_{i\kappa_j} a_+^{-1}(k) = i\mu_j \gamma_{+,j}^2, \quad (2.25)$$

where $\gamma_{\tau,j}^{-2} := \int_{-\infty}^{+\infty} (\varphi(x, t; i\kappa_j))^2 \frac{m+\omega}{\omega} dx > 0$ and $\varphi_{\tau}(x, t; i\kappa_j) = \mu_j \varphi_l(x, t; i\kappa_j)$ with quantities $\gamma_{\tau,j}$, μ_j independent on t .

- $\overline{a^{-1}(-\bar{k})} = a^{-1}(k)$, $k \in \overline{D}$; $\overline{r(-\bar{k})} = r(k)$, $k \in \mathbb{D}_{\mathbb{C}^{\mu}}$;

$$1 - |r(k)|^2 = \frac{z(k)}{k} \frac{1}{|a(k)|^2}, \quad k \in \mathbb{R} \setminus \{0\}.$$

2.2. Vector Riemann – Hilbert problem.

We define the vector Riemann – Hilbert problem as follows. The sectionally meromorphic function $V_{\tau}(y, t; k) = (V_{\tau,1}(y, t; k), V_{\tau,2}(y, t; k))$ is defined by

$$\begin{cases} \sqrt[4]{\frac{m+\omega}{\omega}} \left(\frac{1}{a(k)} \varphi_l(x, t; k) e^{ig_{\tau}(y, t; k)}, \varphi_{\tau}(x, t; k) e^{-ig_{\tau}(y, t; k)} \right), & k \in D, \\ \sqrt[4]{\frac{m+\omega}{\omega}} \left(\overline{\varphi(x, t; \bar{k})} e^{ig_{\tau}(y, t; k)}, \frac{1}{\overline{a(\bar{k})}} \overline{\varphi_{\tau}(x, t; \bar{k})} e^{-ig_{\tau}(y, t; k)} \right), & k \in D^*, \end{cases} \quad (2.26)$$

where $g_{\tau}(y, t; k) = ky - \frac{2\omega kt}{4k^2+1}$, and the domain D is determined by (2.18).

We are interested in the jump relations of $V_{\tau}(y, t; k)$ on the contour

$$\Sigma_{\tau} = \mathbb{R} \cup [i\hat{c}, -i\hat{c}].$$

The orientation of the contour is chosen as follows: from $-\infty$ to $+\infty$ and from $+i\hat{c}$ to $-i\hat{c}$. The positive side of the contour lies on the left as one moves along the contour in the positive direction, the negative one lies on the right. We will use the notation $V_{\pm}(y, t; k)$ for the limit of $V_{\tau}(y, t; k)$ from the positive/negative side of the contour.

The scattering relation (2.23) and Lemma 2.3 provide the following properties of the function $V_{\tau}(y, t; k)$:

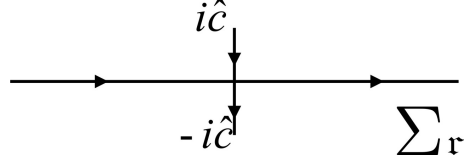


Figure 4: Contour Σ_r .

Lemma 2.4. (cf. [35, Section 3])

1. *Analyticity:*
 $V_r(y, t; k)$ is meromorphic in $k \in \mathbb{C} \setminus \Sigma_r$ with simple poles at $\pm i\kappa_j$ and continuous up to the boundary;
2. *Jump relations:* for $k \in \Sigma_r$
 $V_r^-(y, t; k) = V_r^+(y, t; k)J_r(y, t; k)$, where

$$J_r(y, t; k) = \begin{pmatrix} 1 & \overline{r(k)} e^{-2ig_r(y, t; k)} \\ -r(k) e^{2ig_r(y, t; k)} & 1 - |r(k)|^2 \end{pmatrix}, \quad k \in \mathbb{R} \setminus \{0\}, \quad (2.27)$$

$$J_r(y, t; k) = \begin{pmatrix} 1 & 0 \\ f(k) e^{2ig_r(y, t; k)} & 1 \end{pmatrix}, \quad k \in (i\hat{c}, 0),$$

$$J_r(y, t; k) = \begin{pmatrix} 1 & \overline{f(k)} e^{-2ig_r(y, t; k)} \\ 0 & 1 \end{pmatrix}, \quad k \in (0, -i\hat{c}), \quad (2.28)$$

$$\text{where } r(k) = \frac{b(k)}{a(k)}, \quad f(k) := \frac{z(k+0)}{k a(k-0)a(k+0)}, \quad k \in (i\hat{c}, 0),$$

$$1 - |r(k)|^2 = \frac{z(k)}{k |a(k)|^2}, \quad k \in \mathbb{R};$$

3. *Residue conditions:* for $j = 1, \dots, N$

$$\text{Res}_{i\kappa_j} V_r(y, t; k) = \lim_{k \rightarrow i\kappa_j} V_r(y, t; k) \begin{pmatrix} 0 & 0 \\ i\gamma_{+,j}^2 e^{2ig_r(y, t; i\kappa_j)} & 0 \end{pmatrix}, \quad (2.29a)$$

$$\text{Res}_{-i\kappa_j} V_r(y, t; k) = \lim_{k \rightarrow -i\kappa_j} V_r(y, t; k) \begin{pmatrix} 0 & -i\gamma_{+,j}^2 e^{2ig_r(y, t; i\kappa_j)} \\ 0 & 0 \end{pmatrix} \quad (2.29b)$$

4. *Symmetry relations:*

$$\overline{V_r(y, t; \bar{k})} = V_r(y, t; -k) = V_r(y, t; k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \overline{V_r(y, t; -\bar{k})} = V_r(y, t; k);$$

5. *Asymptotics at infinity:*

$$V_{\tau}(y, t; k) \rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \text{ as } k \rightarrow \infty. \quad (2.30)$$

The solution of the initial value problem (1.1), (1.2) can be obtained via the solution of the RH problem in a parametric form :

Lemma 2.5. (*cf.* [35, Lemma 5.1])

The function $V_{\tau}(x, t; k) = (V_{\tau,1}(x, t; k), V_{\tau,2}(x, t; k))$ defined in (2.26) satisfies the following relations:

$$\frac{V_{\tau,1}(x, t; \frac{i}{2})}{V_{\tau,2}(x, t; \frac{i}{2})} = e^{x-y}, \quad (2.31a)$$

$$V_{\tau,1}(x, t; k)V_{\tau,2}(x, t; k) = \sqrt{\frac{m+\omega}{\omega}} \left(1 + \frac{2i}{\omega} u(x, t) \left(k - \frac{i}{2} \right) + O \left(k - \frac{i}{2} \right)^2 \right), \quad (2.31b)$$

3. Phase functions.

In this section we construct the necessary phase functions and the signature table for their imaginary part. Since the RH problem is formulated in terms of $y(x)$, rather than in terms of x , in the sequel we study the problem in terms of the variables y, t . We study the asymptotics along the rays $y = Ct$, where C is a constant, and hence in asymptotic analysis we use a “slow” variable

$$\xi \equiv \frac{y}{\omega t},$$

and a fast variable t ; hence instead of the phase function $g_{\tau}(y, t; k)$ from (2.26) it is convenient to introduce a phase function which depends only on $\xi \equiv \frac{y}{\omega t}$ and t .

3.1. The right and the left phase function.

As was shown in [35], in asymptotic analysis for large positive x and for large negative x we can use the “right” and the “left” phase functions,

$$g_{\tau}(k, \xi) = k\omega\xi - \frac{2k\omega}{4k^2 + 1}, \quad \text{where } \xi = \frac{y}{\omega t}, \quad (3.32)$$

$$g_{\text{I}}(k, \xi) = \omega\xi\sqrt{k^2 + \hat{c}^2} - \frac{2\omega\sqrt{k^2 + \hat{c}^2}}{(1 - 4\hat{c}^2)(4k^2 + 1)}, \quad (3.33)$$

(here \hat{c} is from (2.11)), which naturally appear in asymptotics for the Jost solutions of the Lax pair. The signature table for $\text{Im}g_{\tau}(k, \xi)$ is drawn in Figures 5, 6. The stationary phase points $\theta'_k(k, \xi) = 0$ are given as $i\mu_{0,\tau}, i\mu_{1,\tau}$, where

$$\mu_{0,\tau}(\xi) = \frac{1}{2}\sqrt{\frac{\xi+1-\sqrt{1+4\xi}}{-\xi}}, \quad \mu_{1,\tau}(\xi) = \frac{1}{2}\sqrt{\frac{\xi+1+\sqrt{1+4\xi}}{-\xi}}. \quad (3.34)$$

The quantity $d_{0,\tau}$ is defined by the formula $d_{0,\tau} = \frac{1}{2}\sqrt{\frac{\xi-2}{\xi}}$.

Once the signature table for $\text{Im}g_{\tau}$ is constructed, the one for $\text{Im}g_{\text{I}}$ is obtained by using the relation $g_{\text{I}}(k, \xi) = \frac{c+\omega}{\omega}g_{\tau}(\hat{z}, \hat{\xi})$, where $\hat{z} = \frac{\sqrt{k^2 + \hat{c}^2}}{\sqrt{1-4\hat{c}^2}}$, $\hat{\xi} = \left(\frac{\omega}{c+\omega}\right)^{3/2}\xi$. Under the conformal mapping $\hat{z} \mapsto k$, the point $\hat{z} = \frac{i}{2}$ comes to the point $k = \frac{i}{2}$ and the segment $\hat{z} \in [-\frac{1}{2}\sqrt{\frac{c}{\omega}}, \frac{1}{2}\sqrt{\frac{c}{\omega}}]$ comes to the interval $k \in [\frac{i}{2}\sqrt{\frac{c}{c+\omega}}, -\frac{i}{2}\sqrt{\frac{c}{c+\omega}}] = [i\hat{c}, -i\hat{c}]$. Therefore, an important role is played by the mutual location of the quantities $\frac{1}{2}\sqrt{\frac{c}{\omega}}$, $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$, and thus, we distinguish the cases

- $\frac{c}{\omega} > 3$,
- $1 < \frac{c}{\omega} < 3$,
- $0 < \frac{c}{\omega} < 1$.

The stationary phase points $\hat{z}_{0,\text{I}}(\hat{\xi}) = \frac{1}{2}\sqrt{\frac{\hat{\xi}+1-\sqrt{1+4\hat{\xi}}}{-\hat{\xi}}}$, $\hat{z}_{1,\text{I}}(\hat{\xi}) = \frac{1}{2}\sqrt{\frac{\hat{\xi}+1+\sqrt{1+4\hat{\xi}}}{-\hat{\xi}}}$ are equal to $\frac{1}{2}\sqrt{\frac{c}{\omega}}$ when $\hat{\xi} = \frac{-2(c-\omega)}{(c+\omega)^2}$. The corresponding quantities in the k -plane are equal to $i\mu_{0,\text{I}}, i\mu_{1,\text{I}}$, where

$$\mu_{0,\text{I}} = \frac{1}{2}\sqrt{\frac{\xi\sqrt{1-4\hat{c}^2}+1-\sqrt{1+4(1-4\hat{c}^2)^{3/2}\xi}}{\xi\sqrt{1-4\hat{c}^2}}}, \quad \mu_{1,\text{I}} = \frac{1}{2}\sqrt{\frac{\xi\sqrt{1-4\hat{c}^2}+1+\sqrt{1+4(1-4\hat{c}^2)^{3/2}\xi}}{\xi\sqrt{1-4\hat{c}^2}}}. \quad (3.35)$$

The signature table for $\text{Im}g_{\text{I}}(k, \xi)$ is drawn in Figures 7–12.

3.2. Middle phase functions.

We use g_{τ} and g_{I} in the asymptotic analysis for large positive and large negative x , when the lines $\text{Im}g_{\tau} = 0$, $\text{Im}g_{\text{I}} = 0$ do not intersect the interval $[i\hat{c}, -i\hat{c}]$. When the above lines start to intersect the above segment, we need to construct intermediate phase functions. The observation

- $g_{\tau,-} - g_{\tau,+} = 0$, and $\text{Im}g_{\tau} > 0$ in a vicinity of the segment $[i\hat{c}, 0]$,
- $g_{\text{I},-} + g_{\text{I},+} = 0$, and $\text{Im}g_{\text{I}} < 0$ in a vicinity of the segment $[i\hat{c}, 0]$

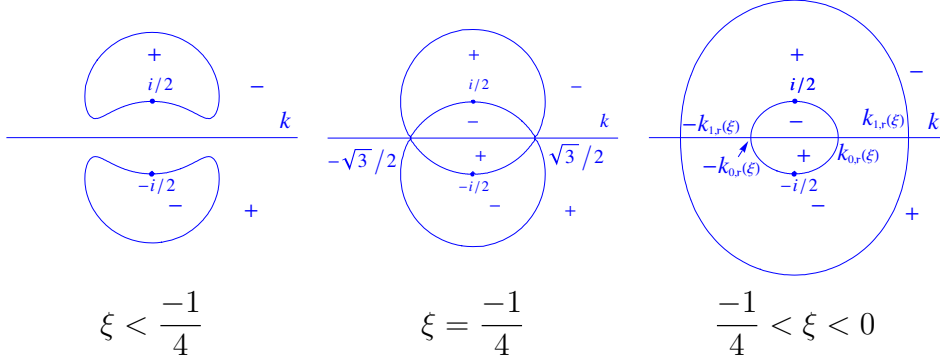


Figure 5: Signature table for $\text{Im} g_{\tau}(k, \xi)$. The plotted contours are the lines $\text{Im} g_{\tau} = 0$.

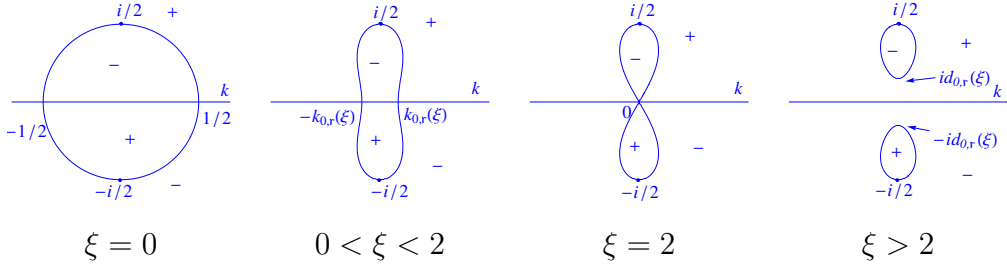


Figure 6: Signature table for $\text{Im} g_{\tau}(k, \xi)$. The plotted contours are the lines $\text{Im} g_{\tau} = 0$.

suggests looking for an (odd) phase function with the following property:

- 1(a). on those subsegments of $(i\hat{c}, 0)$ that lie in the domain $\text{Im} g(k, \xi) \leq 0$, we have

$$g_{-}(k, \xi) + g_{+}(k, \xi) = 0, \quad (3.36)$$

- 1(b). on those subsegments of $(i\hat{c}, 0)$ that lie in the domain $\text{Im} g(k, \xi) \geq 0$, we have

$$g_{-}(k, \xi) - g_{+}(k, \xi) = B(\xi), \quad (3.37)$$

where $B(\xi)$ is a real-valued quantity, which does not depend on k .

Also, a suitable phase function should satisfy the following properties:

2. asymptotics at infinity: $g(k, \xi) - g_{\tau}(k, \xi) = O(k^{-1})$ as $k \rightarrow \infty$; (3.38)

3. asymptotics at $k = \frac{i}{2}$: $\exists \lim_{k \rightarrow i/2} (g(k, \xi) - g_{\tau}(k, \xi)) \in \mathbb{C}$. (3.39)

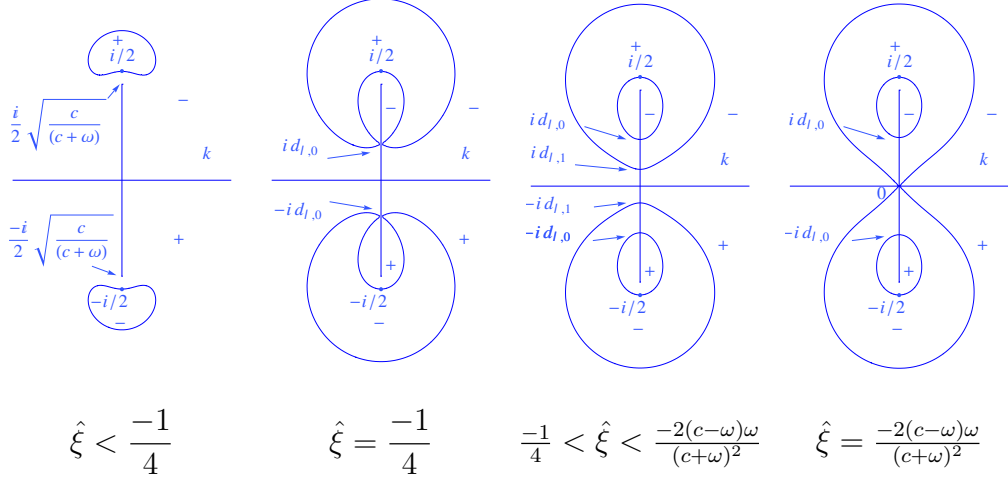


Figure 7: Signature table for $\text{Im} g_l(k, \xi)$ when $\frac{c}{\omega} > 3$. The plotted contours are the lines $\text{Im} g_l = 0$.

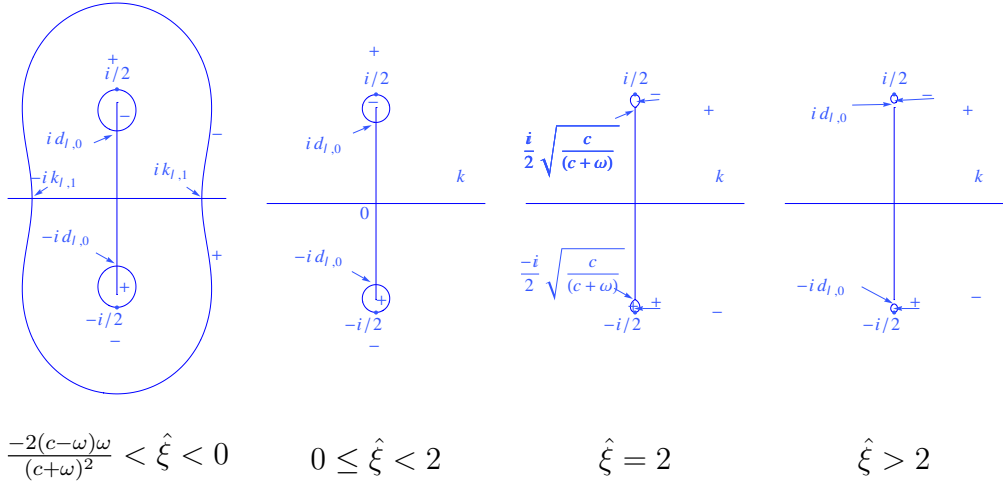


Figure 8: Signature table for $\text{Im} g_l(k, \xi)$ when $\frac{c}{\omega} > 3$. The plotted contours are the lines $\text{Im} g_l = 0$.

Let us notice that $g_r(k, \xi)$ and $g_l(k, \xi)$ satisfy these conditions. Further,

$$dg_r(k, \xi) = dg_l(k, \xi) = \left(\frac{\omega}{4(k \mp \frac{i}{2})^2} + O(1) \right) dk \quad \text{as } k \rightarrow \frac{\pm i}{2}.$$

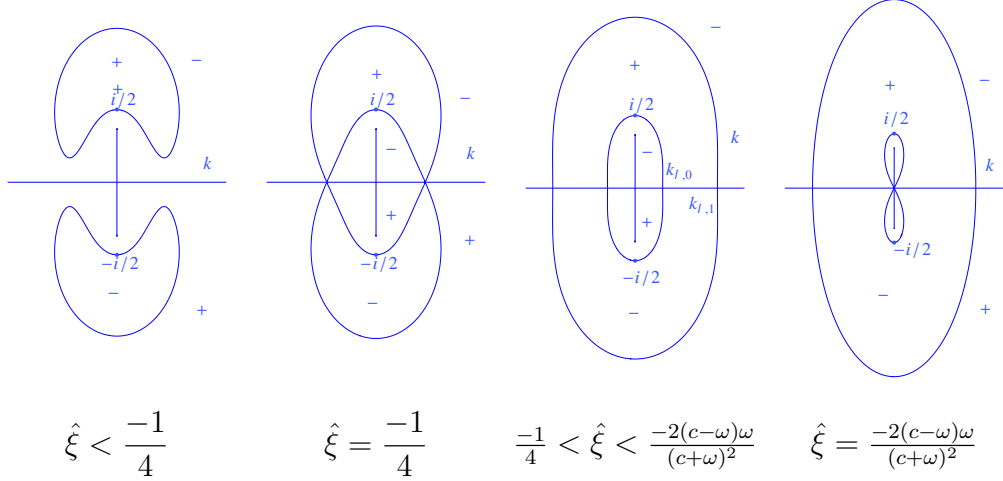


Figure 9: Signature table for $\text{Im} g_l(k, \xi)$ when $1 < \frac{c}{\omega} < 3$. The plotted contours are the lines $\text{Im} g_l = 0$.

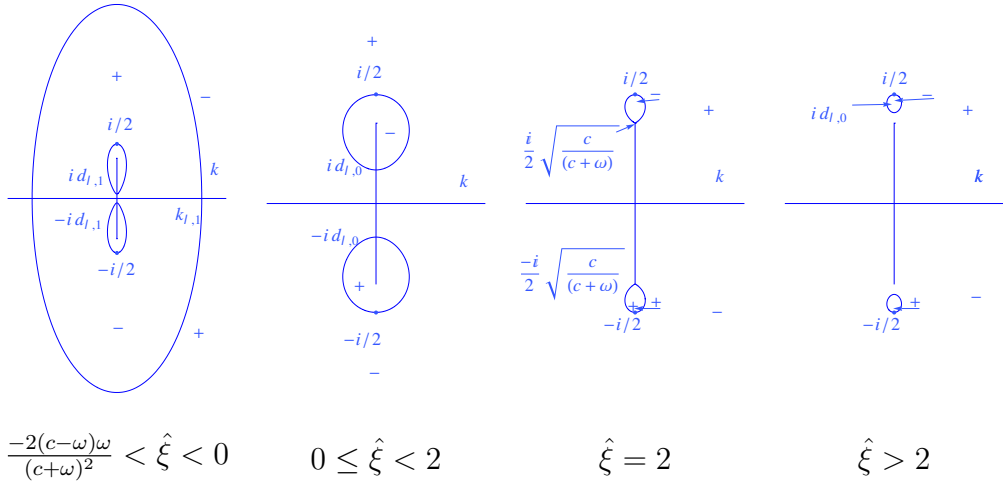


Figure 10: Signature table for $\text{Im} g_l(k, \xi)$ when $1 < \frac{c}{\omega} < 3$. The plotted contours are the lines $\text{Im} g_l = 0$.

The suitable g -functions has qualitatively the same signature table as g_l (see Figures 7–12), but have different properties on the segment $[i\hat{c}, -i\hat{c}]$.

The suitable signature table for g -functions in different regions of the parameter $\frac{c}{\omega}$ and variable ξ is plotted in Figure 13. Here $\xi_j, j = 1, \dots, 5$ are

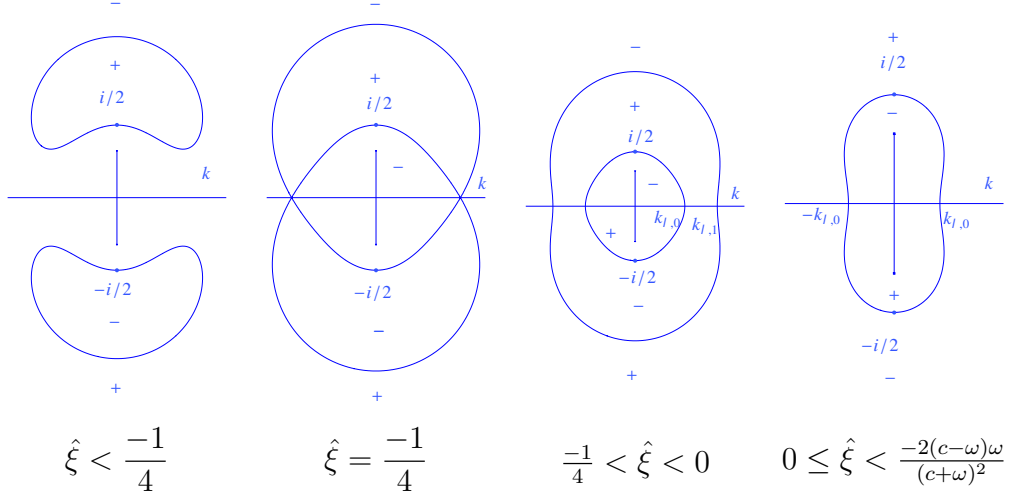


Figure 11: Signature table for $\text{Im} g_l(k, \xi)$ when $0 < \frac{c}{\omega} < 1$. The plotted contours are the lines $\text{Im} g_l = 0$.

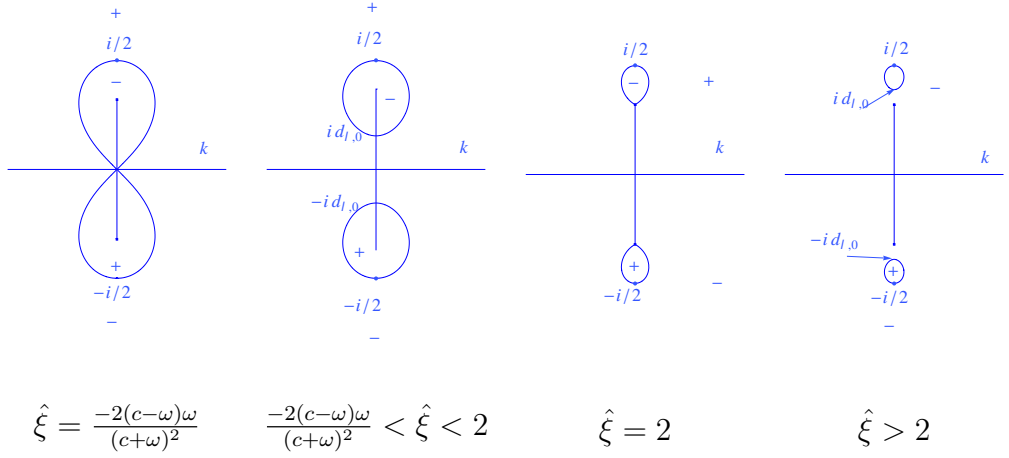


Figure 12: Signature table for $\text{Im} g_l(k, \xi)$ when $0 < \frac{c}{\omega} < 1$. The plotted contours are the lines $\text{Im} g_l = 0$.

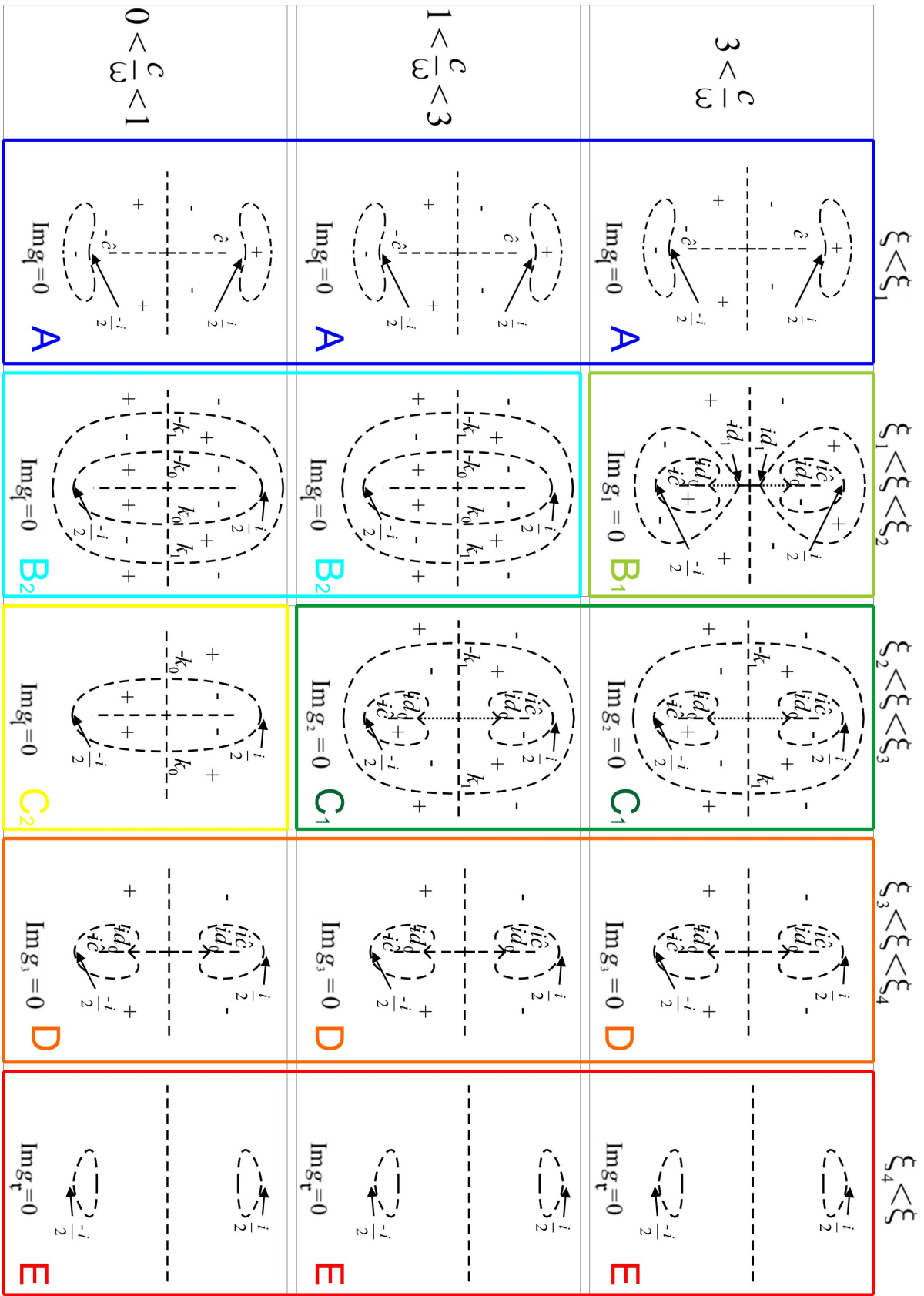


Figure 13: Suitable signature table for $\text{Im}g(k, \xi)$ in different regions of the parameter ξ .

borders between regions with different phase function g . Further, the points $\pm id_0 \in (0, i\hat{c})$, $\pm id_1 \in \pm(0, id_0)$ are the points of intersection of the lines $\text{Im}g = 0$ with the imaginary axis. (The points $\pm id_1$ appear only in the situation plotted in subgraphic B_1 , i.e. $c > 3\omega$, $\xi_1 < \xi < \xi_2$. The points $\pm id_0$ appear in the situation plotted in subgraphics B_1, C_1, D .) Points $\pm k_0, \pm k_1$ are points of intersection of the curve $\text{Im}g = 0$ with the real line (k_0 appears in situation of subgraphics B_2, C_2 , and k_1 appears in situation of subgraphics B_2, C_1 . If both of them are presented in graphic, then $k_0 < k_1$. We can say that the points $\pm id_j, j = 1, 2$ are transformed into the point $\pm k_j, j = 1, 2$ when they move from the imaginary axis to the real axis.)

Let us notice that the corresponding quantities $\xi_j, j = 1, 4$, are different for different values of the parameter $\frac{c}{\omega}$. Denote by ζ_j the corresponding critical values in the $x, t \geq 0$ half-plane, $\zeta = \frac{x}{\omega t}$. We calculate them in Lemma 4.2, but for the sake of brevity, we write them down here. We list in ascending order of the parameter ξ the qualitative description of asymptotics in each zone (we use functions $g_i, i = 1, 2, 3$, defined by (3.42)–(3.45), (3.48)–(3.50)):

• **Case** $\frac{c}{\omega} > 3$. Transition points are: (3.40)

1. $\xi_1 = \frac{-1}{4} \left(\frac{c+\omega}{\omega} \right)^{3/2}, \quad \zeta_1 = \frac{3c-\omega}{4\omega}.$
2. ξ_2 is determined by the transcendental system (3.52).
 $\zeta_2 = \xi_2 - \frac{2i}{\omega} \lim_{k \rightarrow i/2} (g_2 - g_r)(k, \xi_2).$
3. $\xi_3 = 0, \quad \zeta_3 = -\frac{2i}{\omega} \lim_{k \rightarrow i/2} (g_3 - g_r)(k, \xi_3 = 0).$
4. $\xi_4 = 2 \left(\frac{c+\omega}{\omega} \right)^{3/2}, \quad \zeta_4 = \xi_4.$

Therefore, we have a zone of fast decaying to c asymptotics $\xi < \xi_1$, hyperelliptic (genus 2) zone $\xi_1 < \xi < \xi_2$, two zones of elliptic asymptotics $\xi_2 < \xi < \xi_3$ and $\xi_3 < \xi < \xi_4$, and soliton zone $\xi_4 < \xi$.

• **Case** $1 < \frac{c}{\omega} < 3$.

1. $\xi_1 = \frac{-1}{4} \left(\frac{c+\omega}{\omega} \right)^{3/2}, \quad \zeta_1 = \frac{3c-\omega}{4\omega}.$
2. $\xi_2 = \frac{-2(c-\omega)}{\sqrt{\omega(c+\omega)}}, \quad \zeta_2 = \frac{2\omega^2 - c\omega + c^2}{\omega(c+\omega)}.$
3. $\xi_3 = 0, \quad \zeta_3 = -\frac{2i}{\omega} \lim_{k \rightarrow i/2} (g_3 - g_r)(k, \xi_3 = 0).$
4. $\xi_4 = 2 \left(\frac{c+\omega}{\omega} \right)^{3/2}, \quad \zeta_4 = \xi_4.$

Hence, we have a zone of fast decaying to c asymptotics $\xi < \xi_1$, zone of slowly decaying to c dispersive wave $\xi_1 < \xi < \xi_2$, two zones of elliptic asymptotics

$\xi_2 < \xi < \xi_3$ and $\xi_3 < \xi < \xi_4$, and soliton zone $\xi_4 < \xi$.

• **Case** $0 < \frac{c}{\omega} < 1$.

1. $\xi_1 = \frac{-1}{4} \left(\frac{c+\omega}{\omega} \right)^{3/2}$, $\zeta_1 = \frac{3c-\omega}{4\omega}$.
2. $\xi_2 = 0$, $\zeta_2 = \frac{c}{\omega}$.
3. $\xi_3 = \frac{-2(c-\omega)}{\sqrt{\omega(c+\omega)}}$, $\zeta_3 = \frac{2\omega^2 - c\omega + c^2}{\omega(c+\omega)}$.
4. $\xi_4 = 2 \left(\frac{c+\omega}{\omega} \right)^{3/2}$, $\zeta_4 = \xi_4$.

(3.41)

Similarly, we have a zone of fast decaying to c asymptotics $\xi < \xi_1$, two zones of slowly decaying to c dispersive wave $\xi_1 < \xi < \xi_2$ and $\xi_2 < \xi < \xi_3$, zone of elliptic asymptotics $\xi_3 < \xi < \xi_4$, and soliton zone $\xi_4 < \xi$.

In the critical cases $\frac{c}{\omega} = 3$, $\frac{c}{\omega} = 1$, $\frac{c}{\omega} = 0$, some of the zones vanish. Namely,

• **Case** $\frac{c}{\omega} = 3$. Then $\xi_1 = \xi_2 = \frac{-1}{4} \left(\frac{c+\omega}{\omega} \right)^{3/2}$, and we do not have the hyperelliptic zone of genus 2.

• **Case** $\frac{c}{\omega} = 1$. Then $\xi_2 = \xi_3 = 0$, and we do not have the first elliptic zone.

• **Case** $\frac{c}{\omega} = 0$. Then $\xi_3 = \xi_4 = 0$, and we do not have the elliptic zone.

We will try to find a phase function g so, that it would exactly describe the situation shown in Figure 13. For each $i = 1, 2, 3$, define $g = g_i$ as

$$g_i(k, \xi) = \int_{i\hat{c}}^k dg_i(k, \xi), \quad (3.42)$$

where dg_i are chosen as follows for the specified regions (see Figure 13):

1. case B (compare with the 3rd graphic in Figure 7):

$$dg_1(k, \xi) = \frac{\omega \xi (k^2 + \mu^2) k \sqrt{(k^2 + d_0^2)(k^2 + d_1^2)}}{(k^2 + \frac{1}{4})^2 \sqrt{k^2 + \hat{c}^2}} dk, \quad 0 < d_1 < \mu < d_0 < \hat{c}. \quad (3.43)$$

2. case C_1 (compare with the 1st graphic in Figures 8, 10): For $\xi \neq 0$:

$$dg_2(k, \xi) = \frac{\omega \xi (k^2 + \mu_0^2) (k^2 - k_1^2) \sqrt{(k^2 + d_0^2)}}{(k^2 + \frac{1}{4})^2 \sqrt{k^2 + \hat{c}^2}} dk, \quad 0 < \mu_0 < d_0 < \hat{c} < \frac{1}{2}, \quad k_1 > 0. \quad (3.44)$$

3. case D (compare with the 2nd graphic in Figures 8, 10, 12): For $\xi \neq 0$:

$$dg_3(k, \xi) = \frac{\omega \xi (k^2 + \mu_0^2) (k^2 + \mu_1^2) \sqrt{(k^2 + d_0^2)}}{(k^2 + \frac{1}{4})^2 \sqrt{k^2 + \hat{c}^2}} dk, \quad 0 < \mu_0 < d_0 < \hat{c} < \frac{1}{2} < \mu_1. \quad (3.45)$$

Note that if $\xi = 0$, then in (3.44)-(3.45) $k_1 = \infty$, $\mu_1 = \infty$, so we will take care of this case separately. For $\xi = 0$ we set:

$$dg_{2,3}(k, \xi=0) = \frac{\sqrt{1-4\hat{c}^2}}{(1-4\mu_0^2)\sqrt{1-4d_0^2}} \frac{\omega (k^2 + \mu_0^2) \sqrt{k^2 + d_0^2}}{(k^2 + \frac{1}{4})^2 \sqrt{k^2 + \hat{c}^2}} dk, \quad 0 < \mu_0 < d_0 < \hat{c} < \frac{1}{2}. \quad (3.46)$$

Finally, the original right and left phase functions can also be written as

4. $dg_{\mathfrak{r}}(k, \xi) = \frac{\omega \xi (k^2 + \mu_{0,\mathfrak{r}}^2)(k^2 + \mu_{1,\mathfrak{r}}^2)}{(k^2 + \frac{1}{4})^2} dk$ with $\mu_{0,\mathfrak{r}}, \mu_{1,\mathfrak{r}}$ defined in (3.34).
5. $dg_{\mathfrak{l}}(k, \xi) = \frac{\omega \xi k (k^2 + \mu_{0,\mathfrak{l}}^2)(k^2 + \mu_{1,\mathfrak{l}}^2)}{(k^2 + \frac{1}{4})^2 \sqrt{k^2 + \hat{c}^2}} dk$ with $\mu_{0,\mathfrak{l}}, \mu_{1,\mathfrak{l}}$ defined in (3.35).

3.3. Equations for the parameters of the middle phase function

All of the functions (3.43)-(3.45) must satisfy properties (3.36)-(3.39), and this leads to systems of equations that determine the parameters $\mu_0, \mu_1, d_0, d_1, k_1$ of the g -functions. Property (3.38) is satisfied automatically. To satisfy (3.36)-(3.37) it is enough to satisfy

$$\int_{id_0}^{id_1} dg_1 = \int_0^{id_0} dg_2 = \int_0^{id_0} dg_3 = 0. \quad (3.47)$$

Finally, to satisfy (3.39), let us expand dg_i in a neighborhood of $k = i/2$:

$$i = 1 : \frac{dg_1}{dk} = \frac{-\xi \omega \sqrt{1 - 4d_0^2} \sqrt{1 - 4d_1^2} (1 - 4\mu_0^2)}{16\sqrt{1 - 4\hat{c}^2} \left(k \mp \frac{i}{2}\right)^2} \mp \frac{i\xi \omega [(1 - 4d_0^2)(1 - 4d_1^2)(1 - 4\mu_0^2 - 2(1 - 4\hat{c}^2)) - (1 - 4\hat{c}^2)(1 - 4\mu_0^2)(1 - 4d_0^2 + 1 - 4d_1^2)]}{8\sqrt{1 - 4d_0^2} \sqrt{1 - 4d_1^2} (1 - 4\hat{c}^2)^{3/2} \left(k \mp \frac{i}{2}\right)} + O(1),$$

Correspondingly, the equations for the parameters are:

$$\frac{-\xi \omega \sqrt{1 - 4d_0^2} \sqrt{1 - 4d_1^2} (1 - 4\mu_0^2)}{16\sqrt{1 - 4\hat{c}^2}} = \frac{1}{4}, \quad (3.48a)$$

$$1 - 4\mu_0^2 = \frac{2(1 - 4\hat{c}^2)(1 - 4d_0^2)(1 - 4d_1^2)}{4(\hat{c}^2 - d_1^2)(1 - 4d_0^2) - (1 - 4\hat{c}^2)(1 - 4d_1^2)}, \quad (3.48b)$$

$$\int_{id_0}^{id_1} \frac{k(k^2 + \mu_0^2) \sqrt{k^2 + d_0^2} \sqrt{k^2 + d_1^2}}{(k^2 + \frac{1}{4})^2 \sqrt{k^2 + \hat{c}^2}} = 0. \quad (3.48c)$$

$$i = 2 : \frac{dg_2}{dk} = \frac{-\xi \omega \sqrt{1 - 4d_0^2} (4k_1^2 + 1) (1 - 4\mu_0^2)}{16\sqrt{1 - 4\hat{c}^2} \left(k \mp \frac{i}{2}\right)^2} \mp \frac{i\xi \omega \left[\left\{4\hat{c}^2 + 4k_1^2 + (1 - 4\hat{c}^2)4k_1^2\right\}(1 - 4d_0^2) - (1 - 4\hat{c}^2)(1 + 4k_1^2)\right](1 - 4\mu_0^2) - 2(1 - 4\hat{c}^2)(1 - 4d_0^2)(1 + 4k_1^2)}{8\sqrt{1 - 4d_0^2} (1 - 4\hat{c}^2)^{3/2} \left(k \mp \frac{i}{2}\right)}$$

+O(1).

Correspondingly, the equations for the parameters are:

$$\frac{-\xi\sqrt{1-4d_0^2}(4k_1^2+1)(1-4\mu_0^2)}{16\sqrt{1-4\hat{c}^2}} = \frac{1}{4}, \quad (3.49a)$$

$$1-4\mu_0^2 = \frac{2(1-4\hat{c}^2)(1-4d_0^2)(1+4k_1^2)}{\{4\hat{c}^2+4k_1^2+(1-4\hat{c}^2)4k_1^2\}(1-4d_0^2)-(1-4\hat{c}^2)(1+4k_1^2)}, \quad (3.49b)$$

$$\int_0^{id_0} \frac{(k^2-k_1^2)(k^2+\mu_0^2)\sqrt{k^2+d_0^2}}{(k^2+\frac{1}{4})^2\sqrt{k^2+\hat{c}^2}} dk = 0. \quad (3.49c)$$

$$i = 3 : \frac{dg_3}{dk} = \frac{\xi\omega\sqrt{1-4d_0^2}(4\mu_1^2-1)(1-4\mu_0^2)}{16\sqrt{1-4\hat{c}^2}(k \mp \frac{i}{2})^2} \pm \frac{i\xi\omega\sqrt{1-4\hat{c}^2}[(1-4\mu_0^2)(1-4\mu_1^2)(2(1-4d_0^2)-1)-2(1-4\mu_0^2+1-4\mu_1^2)(1-4d_0^2)]}{8\sqrt{1-4d_0^2}(1-4\hat{c}^2)(k \mp \frac{i}{2})} + O(1).$$

Correspondingly, the equations for the parameters are:

$$\frac{\xi\sqrt{1-4d_0^2}(4\mu_1^2-1)(1-4\mu_0^2)}{16\sqrt{1-4\hat{c}^2}} = \frac{1}{4}, \quad (3.50a)$$

$$1-4\mu_0^2 = \frac{2(1-4\hat{c}^2)(1-4d_0^2)(4\mu_1^2-1)}{[4\mu_1^2-1+(1-4\hat{c}^2)(4\mu_1^2+1)](1-4d_0^2)-(1-4\hat{c}^2)(4\mu_1^2-1)}, \quad (3.50b)$$

$$\int_0^{id_0} \frac{(k^2+\mu_0^2)(k^2+\mu_1^2)\sqrt{k^2+d_0^2}dk}{(k^2+\frac{1}{4})^2\sqrt{k^2+\hat{c}^2}} = 0. \quad (3.50c)$$

For $\xi = 0$ the equations for the parameters of the function

$g_2(k, \xi = 0) \equiv g_3(k, \xi = 0)$ defined by (3.46) are

$$1-4\mu_0^2 = \frac{2(1-4\hat{c}^2)(1-4d_0^2)}{(2-4\hat{c}^2)(1-4d_0^2)-(1-4\hat{c}^2)}, \quad (3.51a)$$

$$\int_0^{id_0} \frac{(k^2+\mu_0^2)\sqrt{k^2+d_0^2}dk}{(k^2+\frac{1}{4})^2\sqrt{k^2+\hat{c}^2}} = 0. \quad (3.51b)$$

For the initial phase functions we have

$$\text{Case of } g_{\tau} : \frac{dg_{\tau}(k, \xi)}{dk} = \frac{\xi \omega (1 - 4\mu_0^2) (4\mu_1^2 - 1)}{16 (k \mp \frac{i}{2})^2} \mp \frac{i\xi \omega [(1 - 4\mu_0^2)(1 - 4\mu_1^2) - 2(1 - 4\mu_0^2 + 1 - 4\mu_1^2)]}{8 (k \mp \frac{i}{2})} + O(1).$$

Here $\mu_0 = \mu_{0,\tau}$, $\mu_1 = \mu_{1,\tau}$ are defined in (3.34).

Correspondingly, the equations for the parameters are

$$\begin{cases} \frac{\xi (1 - 4\mu_0^2) (4\mu_1^2 - 1)}{16} = \frac{1}{4}, \\ 1 - 4\mu_0^2 = \frac{2(4\mu_1^2 - 1)}{4\mu_1^2 + 1}. \end{cases}$$

$$\text{Case of } g_l : \frac{dg_l(k, \xi)}{dk} = \frac{\xi \omega (1 - 4\mu_0^2) (4\mu_1^2 - 1)}{16\sqrt{1 - 4\hat{c}^2} (k \mp \frac{i}{2})^2} \mp \frac{i\xi \omega [(1 - 4\mu_0^2)(1 - 4\mu_1^2) - 2(1 - 4\hat{c}^2)(1 - 4\mu_0^2 + 1 - 4\mu_1^2)]}{8(1 - 4\hat{c}^2)^{3/2} (k \mp \frac{i}{2})} + O(1).$$

Here $\mu_0 = \mu_{0,l}$, $\mu_1 = \mu_{1,l}$ are defined in (3.35).

Correspondingly, equations for the parameters are

$$\begin{cases} \frac{\xi (1 - 4\mu_0^2) (4\mu_1^2 - 1)}{16\sqrt{1 - 4\hat{c}^2}} = \frac{1}{4}, \\ 1 - 4\mu_0^2 = \frac{2(1 - 4\hat{c}^2)(4\mu_1^2 - 1)}{2(1 - 4\hat{c}^2) + 4\mu_1^2 - 1}. \end{cases}$$

3.4. Transitions between regions with different phase functions

Now let us take a look at the values $\xi_j, j = 1, 2, 3, 4$, which separate regions of different phase functions. These values depend on whether $\frac{c}{\omega}$ lies in the interval $(3, \infty)$, $(1, 3)$, or $(0, 1)$. The analysis of systems (3.52), (3.53) is presented in subsection 3.5.

Case 1. Transition from $\{\xi < \xi_1\}$ to $\{\xi_1 < \xi < \xi_2\}$ in the case $c/\omega > 3$.

This case is characterized by changing of the phase function $g_l(k, \xi)$ with $g_1(k, \xi)$; it occurs when $d_0 = d_1 = \mu_0$. Equations (3.48) give us $1 - 4d_0^2 = 4(1 - 4\hat{c}^2)$ and $\xi_1 = \frac{-1}{4} \left(\frac{c}{\omega} + 1\right)^{3/2}$.

Case 2. Transition from $\{\xi_1 < \xi < \xi_2\}$ to $\{\xi_2 < \xi < \xi_3\}$ in the case $c/\omega > 3$.

This case is characterized by changing of the phase function $g_1(k, \xi)$ with

$g_2(k, \xi)$; it occurs when $d_1 = 0$. Equations (3.49) give us

$$\frac{-\xi_2 \sqrt{1 - 4d_0^2} (1 - 4\mu_0^2)}{4\sqrt{1 - 4\hat{c}^2}} = 1, \quad (3.52a)$$

$$1 - 4\mu_0^2 = \frac{2(1 - 4\hat{c}^2)(1 - 4d_0^2)}{4\hat{c}^2(1 - 4d_0^2) - (1 - 4\hat{c}^2)}, \quad (3.52b)$$

$$\int_{id_0}^0 \frac{k^2(k^2 + \mu_0^2)\sqrt{k^2 + d_0^2}}{(k^2 + \frac{1}{4})^2 \sqrt{k^2 + \hat{c}^2}} = 0. \quad (3.52c)$$

Case 3. Transition from $\{\xi_2 < \xi < \xi_3\}$ to $\{\xi_3 < \xi < \xi_4\}$ in the case $c/\omega > 1$. This case is characterized by changing of the phase function $g_2(k, \xi)$ with $g_3(k, \xi)$; it occurs when $k_1 = \infty$. Equations (3.49) give us

$$\xi_3 = 0, \quad (3.53a)$$

$$1 - 4\mu_0^2 = \frac{2(1 - 4\hat{c}^2)(1 - 4d_0^2)}{(1 + 1 - 4\hat{c}^2)(1 - 4d_0^2) - (1 - 4\hat{c}^2)}, \quad (3.53b)$$

$$\int_0^{id_0} \frac{(k^2 + \mu_0^2)\sqrt{k^2 + d_0^2}}{(k^2 + \frac{1}{4})^2 \sqrt{k^2 + \hat{c}^2}} = 0. \quad (3.53c)$$

3.5. Unique solvability of the systems for the parameters of g -functions.

In this subsection we discuss the solvability of the above systems (3.48), (3.49), (3.50), (3.52), (3.53), for the parameters of the phase functions.

Lemma 3.1. *Each of the systems (3.48), (3.49), (3.50), (3.52), (3.53) has a unique solution that satisfies $0 \leq d_1 \leq \mu_0 \leq d_0 < \hat{c} < \frac{1}{2}$.*

Proof. Let us treat, for instance, systems (3.52) and (3.48), the other ones can be treated in a similar way.

• Consider first system (3.52). Equation (3.52b) determines $1 - 4\mu_0^2$ as a decreasing function of $1 - 4d_0^2$, which has a vertical asymptote at

$$1 - 4d_0^2 = \frac{1 - 4\hat{c}^2}{4\hat{c}^2} = \frac{\omega}{c}, \quad (\text{see Figure 14, left}) \text{ and we have that}$$

$$1 - 4\mu_0^2 = \frac{2}{\frac{c}{\omega} - 1} \text{ as } 1 - 4d_0^2 = 1.$$

On the other hand, in the same manner as in [32], we can show that equation (3.52c) determines μ_0 as an increasing function of d_0 , or $1 - 4\mu_0^2$ as an increasing function of $1 - 4d_0^2$, with

$$1 - 4\mu_0^2 = 1 \text{ as } 1 - 4d_0^2 = 1.$$

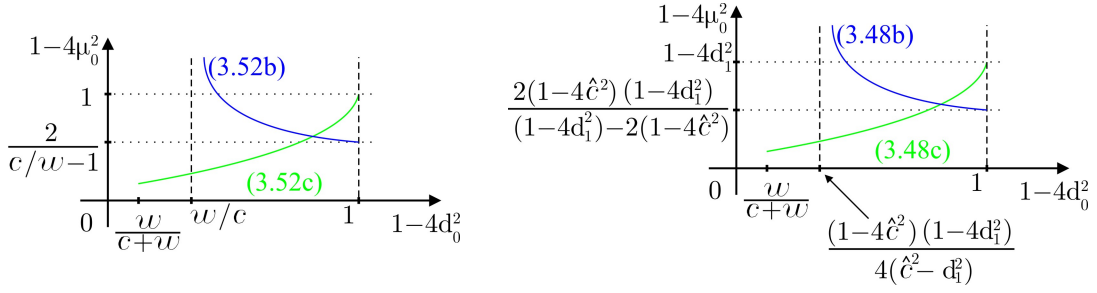


Figure 14: Chart of system (3.52b)-(3.52c) (left) and (3.48b)-(3.48c) (right).

We have: $1 - 4\mu_0^2 = \frac{2}{\frac{c}{w}-1} < 1$ provided $\frac{c}{w} > 3$, hence, (3.52b)–(3.52c) determine a unique pair $\mu_0 \leq d_0$, where the equality is attained only when $\mu_0 = d_0 = 0$. Further, substituting those μ_0, d_0 in (3.52a), we determine $\xi_2 < 0$.

• Now consider system (3.48). First we consider system (3.48b)–(3.48c) as a system of variables d_0, μ_0 with a parameter d_1 . Equation (3.48b) determines $1 - 4\mu_0^2$ as a decreasing function of $1 - 4d_0^2$, which has a vertical asymptote at

$$1 - 4d_0^2 = \frac{(1 - 4\hat{c}^2)(1 - 4d_1^2)}{4(\hat{c}^2 - d_1^2)}, \quad (\text{see Figure 14, right}) \text{ and we have that}$$

$$1 - 4\mu_0^2 = \frac{2(1 - 4\hat{c}^2)(1 - 4d_1^2)}{(1 - 4d_1^2) - 2(1 - 4\hat{c}^2)} \text{ as } 1 - 4d_0^2 = 1 - 4d_1^2.$$

On the other hand, in the same manner as in [32], we can show that equation (3.48c) determines μ_0 as an increasing function of d_0 , or $1 - 4\mu_0^2$ as an increasing function of $1 - 4d_0^2$, with

$$1 - 4\mu_0^2 = 1 - 4d_1^2 \text{ as } 1 - 4d_0^2 = 1 - 4d_1^2.$$

We have

$$\frac{(1 - 4\hat{c}^2)(1 - 4d_1^2)}{4(\hat{c}^2 - d_1^2)} < 1 - 4d_1^2 \text{ and } \frac{2(1 - 4\hat{c}^2)(1 - 4d_1^2)}{(1 - 4d_1^2) - 2(1 - 4\hat{c}^2)} < 1 - 4d_1^2$$

provided $4(1 - 4\hat{c}^2) < 1 - 4d_1^2$.

If the latter inequality is satisfied, then (3.48b)–(3.48c) determine a unique pair $\mu_0 \leq d_0$, where the equality is attained only when

$$d_1 = \mu_0 = d_0 \quad \text{and} \quad 4(1 - 4\hat{c}^2) = 1 - 4d_1^2.$$

Further, substituting those μ_0, d_0 in (3.48a), we determine d_1 , which satisfies $0 \leq d_1 < \mu_0$ by virtue of (3.48c).

Hence, the only thing which remains to do is to establish the inequality

$$4(1 - 4\hat{c}^2) < 1 - 4d_1^2$$

inside of the zone. It is satisfied in the critical case when $d_1 = 0$. Suppose the contrary, $4(1 - 4\hat{c}^2) = 1 - 4d_1^2$. In this case (3.48b) can be rewritten as

$$(1 - 4\mu_0^2) [(1 - 4d_0^2) - (1 - 4d_1^2)] + 2(1 - 4d_0^2) [(1 - 4\mu_0^2) - (1 - 4d_1^2)] = 0,$$

and since $d_1 \leq \mu_0 \leq d_0$, we obtain $d_1 = \mu_0 = d_0$, and hence we have the opening of the hyperelliptic zone. This contradiction shows that inside of the zone we have $4(1 - 4\hat{c}^2) < 1 - 4d_1^2$. \square

4. Riemann-Hilbert problem transformations

In this section we present a series of transformations of the original RH problem from Lemma 2.4 in the domains $\xi_1 < \xi < \xi_4$ in the spirit of the nonlinear steepest descent method [23], which leads to either trivial model problems or model problems explicitly solvable in terms of elliptic (genus 1) or hyperelliptic (genus 2) functions.

Step 1.

First, we exchange the phase function $g_{\tau}(y, t; k)$ with g -function $g(k, \xi)$ constructed in Section 3:

$$V_{\tau}^{(1)}(\xi, t, k) = V_{\tau}(\xi, t, k) e^{i(tg(k, \xi) - g_{\tau}(y, t; k))\sigma_3}.$$

For the brevity of exposition we keep the same letter g for all the cases of variable $\frac{c}{\omega}$, ξ shown in Figure 13. The actual meaning of $g(k, \xi)$ is as follows:

- in cases A, B_2, C_2 we take $g(k, \xi) = g_1(k, \xi)$;
- in case B_1 we take $g(k, \xi) = g_1(k, \xi)$;
- in case C_1 we take $g(k, \xi) = g_2(k, \xi)$;
- in case D we take $g(k, \xi) = g_3(k, \xi)$;
- in case E we take $g(k, \xi) = g_{\tau}(k, \xi)$.

Then the jump matrix and pole conditions for $V_{\tau}^{(1)}$ have the same form as for V_{τ} with the phase function being changed from $g_{\tau}(y, t; k)$ to $tg(k, \xi)$ everywhere in formulas (2.27)-(2.30); the jump matrices on the intervals $(i\hat{c}, -i\hat{c})$ are read now as follows:

$$J_{\tau}(x, t; k) = \begin{pmatrix} e^{it(g_{-}(k, \xi) - g_{+}(k, \xi))} & 0 \\ f(k) e^{it(g_{-}(k, \xi) + g_{+}(k, \xi))} & e^{-it(g_{-}(k, \xi) - g_{+}(k, \xi))} \end{pmatrix}, \quad k \in (i\hat{c}, 0);$$

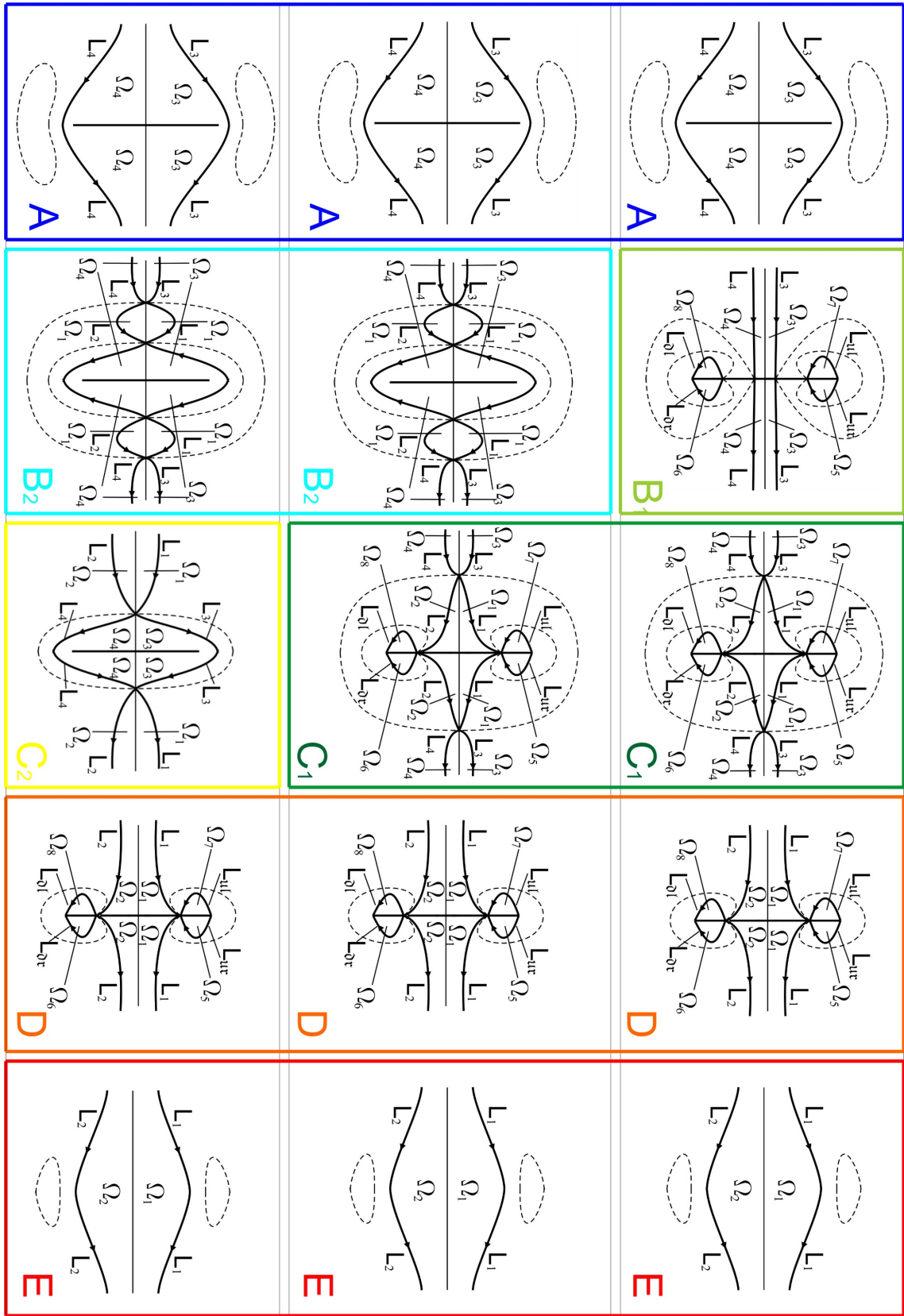


Figure 15: Contour transformations for the RH problem in different regions of the parameter ξ .

$$J_{\mathbf{r}}(x, t; k) = \begin{pmatrix} e^{it(g_-(k, \xi) - g_+(k, \xi))} & \overline{f(\bar{k})} e^{-it(g_-(k, \xi) + g_+(k, \xi))} \\ 0 & e^{-it(g_-(k, \xi) - g_+(k, \xi))} \end{pmatrix}, \quad k \in (0, -i\hat{c}); \quad (4.54)$$

Step 2. Regular Riemann–Hilbert problem.

Following ideas from [10], we transform our meromorphic RH problem into a holomorphic one. In order to achieve this, we define the function $V_{\mathbf{r}}^{(2)}(x, t; k)$ as follows:

$$V_{\mathbf{r}}^{(2)}(x, t; k) = \begin{cases} V_{\mathbf{r}}^{(1)}(x, t; k) \begin{pmatrix} 1 & 0 \\ -i\gamma_{+,j}^2 e^{2itg(\xi, i\kappa_j)} & 1 \end{pmatrix}, & |k - i\kappa_j| < \varepsilon, \\ V_{\mathbf{r}}^{(1)}(x, t; k) \begin{pmatrix} 1 & i\gamma_{+,j}^2 e^{2itg(\xi, i\kappa_j)} \\ 0 & k + i\kappa_j \end{pmatrix}, & |k + i\kappa_j| < \varepsilon, \\ V_{\mathbf{r}}^{(1)}(x, t; k), & \text{elsewhere,} \end{cases} \quad (4.55)$$

where $\varepsilon > 0$ is a sufficiently small number such that circles $|k - i\kappa_j| = \varepsilon$ do not intersect and lie in the domain $\text{Im}z(k) > 0$.

The vector-valued function $V_{\mathbf{r}}^{(2)}(x, t; k)$ solves the following RH problem: find a sectionally holomorphic function $V_{\mathbf{r}}^{(2)}(x, t; k)$ which satisfies the following conditions:

1. $V_{\mathbf{r}}^{(2)}(x, t; \cdot)$ is holomorphic away from the contour $\Sigma_{\mathbf{r}}$ and circles $C_j := \{k : |k - i\kappa_j| = \varepsilon\}$, $\overline{C}_j := \{k : |k + i\kappa_j| = \varepsilon\}$. The orientation on C_j , \overline{C}_j is counterclockwise, so the positive side is inside the circles. It satisfies the same symmetry conditions and has the same asymptotics for large k as $V^{(1)}(\xi, t; k)$;
2. The jump condition $V_{\mathbf{r},-}^{(2)}(\xi, t; k) = V_{\mathbf{r},+}^{(2)}(\xi, t; k)J_{\mathbf{r}}^{(2)}(\xi, t; k)$ is satisfied, where

$$J_{\mathbf{r}}^{(2)}(\xi, t; k) \equiv J_{\mathbf{r}}^{(1)}(\xi, t; k), \quad k \in \Sigma_{\mathbf{r}},$$

$$J_{\mathbf{r}}^{(2)}(\xi, t; k) = \begin{pmatrix} 1 & 0 \\ i\gamma_{+,j}^2 e^{2itg(i\kappa_j, \xi)} & 1 \end{pmatrix}, \quad k \in C_j; \quad (4.56)$$

$$J_{\tau}^{(2)}(\xi, t; k) = \begin{pmatrix} 1 & \frac{-i\gamma_{+,j}^2 e^{2itg(i\kappa_j, \xi)}}{k + i\kappa_j} \\ 0 & 1 \end{pmatrix}, \quad k \in \overline{C}_j. \quad (4.57)$$

Step 3. Following [10], to deal with poles we introduce the function

$$\Lambda(k) = \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}$$

and the following transformation of the RH problem, which acts inside the disks surrounding the poles: $V_{\tau}^{(3)}(x, t; k) = V_{\tau}^{(2)}(x, t; k)D(k)$, where

$$D(k) = \begin{cases} \begin{pmatrix} 1 & \frac{k - i\kappa_j}{-i\gamma_{+,j}^2 e^{2itg(i\kappa_j, \xi)}} \\ \frac{i\gamma_{+,j}^2 e^{2itg(i\kappa_j, \xi)}}{k - i\kappa_j} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\Lambda(k)} & 0 \\ 0 & \Lambda(k) \end{pmatrix}, & |k - i\kappa_j| < \varepsilon, \\ \begin{pmatrix} 0 & \frac{-i\gamma_{+,j}^2 e^{2itg(i\kappa_j, \xi)}}{k + i\kappa_j} \\ \frac{k + i\kappa_j}{i\gamma_{+,j}^2 e^{2itg(i\kappa_j, \xi)}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\Lambda(k, \xi)} & 0 \\ 0 & \Lambda(k, \xi) \end{pmatrix}, & |k + i\kappa_j| < \varepsilon, \\ \Lambda^{-\sigma_3}(k, \xi), & \text{elsewhere.} \end{cases}$$

It is easy to notice that $D(k)$ is a piecewise holomorphic function and does not have poles at the points $\pm i\kappa_j$, $j = 1, \dots, N$.

Now the jump matrix $J_{\tau}^{(3)}(x, t; k) = (D_+(k, \xi))^{-1} J_{\tau}^{(2)}(x, t; k) D_-(k, \xi)$

$$\begin{aligned} J_{\tau}^{(3)}(x, t; k) &= \begin{pmatrix} 1 & \frac{(k - i\kappa_j)\Lambda^2(k)}{i\gamma_{+,j}^2 e^{2itg(i\kappa_j, \xi)}} \\ 0 & 1 \end{pmatrix}, \quad |k - i\kappa_j| = \varepsilon, \\ &= \begin{pmatrix} 1 & 0 \\ \frac{(k + i\kappa_j)\Lambda^{-2}(k)}{-i\gamma_{+,j}^2 e^{2itg(i\kappa_j, \xi)}} & 1 \end{pmatrix}, \quad |k + i\kappa_j| = \varepsilon, \end{aligned}$$

on the circles around the points $i\kappa_j$ is exponentially small for large t , and therefore, they do not contribute in the leading term of the asymptotics.

We recall (3.36)-(3.37) that

$$g_- - g_+ = B \text{ for } k \in (id_0, id_1) \cup (-id_1, -id_0), \quad \text{and}$$

$$g_- + g_+ = 0 \text{ for } k \in (i\hat{c}, id_0) \cup (-id_0, -i\hat{c}) \cup (id_1, -id_1).$$

This holds for all the regions where the above quantities have sense. In the case when d_1 is not defined, we substitute it with 0 in the above relations (see also Figure 13).

Step 4.

On different parts of the real line we need a lower/upper or an upper/lower triangular factorization of the jump matrix. Thee lower/ upper triangular factorization is

$$J^{(3)} = \begin{pmatrix} 1 & 0 \\ -r(k)\Lambda^{-2}(k)e^{2itg} & 1 \end{pmatrix} \begin{pmatrix} 1 & \overline{r(k)}\Lambda^2(k)e^{-2itg} \\ 0 & 1 \end{pmatrix}.$$

To provide an upper/lower triangular factorization of the jump matrix supported on the real line, we introduce the δ - transformation: a new function

$$V^{(4)}(\xi, t; k) = V^{(3)}(\xi, t; k)\delta^{-\sigma_3}$$

has a jump matrix

$$J^{(4)} = \delta_+^{\sigma_3} J^{(3)} \delta_-^{-\sigma_3}.$$

Then on the real axis

$$J^{(4)}(\xi, t; k) = \begin{pmatrix} 1 & \frac{k}{z(k)}a(k)\overline{b(k)}\delta_+^2(k)\Lambda^2(k)e^{-2itg} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-k}{z(k)}b(k)a(\overline{k})\Lambda^{-2}(k)\delta_-^{-2}(k)e^{2itg} & 1 \end{pmatrix},$$

providing

$$\frac{\delta_+}{\delta_-} = 1 - |r(k)|^2, \quad k \in (-\infty, -k_1) \cup (-k_0, k_0) \cup (k_1, \infty),$$

where (we refer to $A, B_j, C_j, D, E, j = 1, 2$, as to subgraphics of Figure 13)

$$k_1 = \begin{cases} 0, & \text{for } A, B_1, \\ k_1(\xi), & \text{for } B_2, C_1, \\ +\infty, & \text{for } C_2, D, E, \end{cases} \quad \text{and } k_0 = \begin{cases} 0, & \text{for } A, B_1, C_1, D, E, \\ k_0(\xi), & \text{for } B_2, C_2. \end{cases}$$

In the cases D, E , this δ -transformation is trivial, $\delta(k) \equiv 1$. For $k_1 \in (0, +\infty)$ we have

$$\delta(k, \xi) = \exp \left(\frac{1}{2\pi i} \left\{ \int_{-\infty}^{-k_1(\xi)} + \int_{-k_0(\xi)}^{+k_0(\xi)} + \int_{k_1(\xi)}^{+\infty} \right\} \frac{\log(1 - |r(s)|^2) ds}{s - k} \right). \quad (4.58)$$

The jump matrix on the interval $(i\hat{c}, -i\hat{c})$ is as follows:

$$\begin{aligned}
J^{(4)}(\xi, t; k) &= \begin{pmatrix} \frac{\delta_+}{\delta_-} e^{it(g_- - g_+)} & 0 \\ \frac{f(k)e^{it(g_- + g_+)}}{\Lambda^2(k)\delta_+\delta_-} & \frac{\delta_-}{\delta_+} e^{-it(g_- - g_+)} \end{pmatrix}, k \in (i\hat{c}, 0), \\
&= \begin{pmatrix} \frac{\delta_+}{\delta_-} e^{it(g_- - g_+)} & \overline{f(\bar{k})}\Lambda^2(k)\delta_+\delta_- e^{-it(g_- + g_+)} \\ 0 & e^{-it(g_- - g_+)} \frac{\delta_-}{\delta_+} \end{pmatrix}, k \in (0, -i\hat{c}).
\end{aligned} \tag{4.59}$$

Step 5.

Here we deal with oscillation behavior of jump matrix on the real line. Using triangular factorizations provided in Step 4, we remove oscillations from \mathbb{R} , moving exponentials from jump matrices to those domains, where they are small. The function $g(k, \xi)$ and its imaginary part (see the signature table of $\text{Im}g(k, \xi)$ at Figure 13 on p. 19) suggest a choice of a new contour Σ_2 for RH problem (see Figure 15 on p. 28). We use the standard lower-upper and upper-lower factorization of the jump matrix on the real axis (see for example [23], formulas (0.11) and (0.23)) and apply the transformation

$$V^{(5)}(\xi, t, k) = V^{(4)}(\xi, t, k)G^{(5)}(\xi, t, k), \text{ where}$$

$$G^{(5)}(\xi, t, k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -r(k)\Lambda^{-2}(k)\delta^{-2}(k)e^{2itg(k, \xi)} & 1 \end{pmatrix}, & k \in \Omega_1, \\ \begin{pmatrix} 1 & -r(\bar{k})\Lambda^2(k)\delta^2(k)e^{-2itg(k)} \\ 0 & 1 \end{pmatrix}, & k \in \Omega_2, \\ \begin{pmatrix} 1 & \frac{k}{z(k)}a(k)\overline{b(\bar{k})}\delta^2(k)\Lambda^2(k)e^{-2itg} \\ 0 & 1 \end{pmatrix}, & k \in \Omega_3, \\ \begin{pmatrix} 1 & 0 \\ \frac{k}{z(k)}b(k)\overline{a(\bar{k})}\Lambda^{-2}(k)\delta^{-2}(k)e^{2itg} & 1 \end{pmatrix}, & k \in \Omega_4, \\ I, & \text{elsewhere,} \end{cases} \tag{4.60}$$

where the domains $\Omega_j, j = 1, 2, 3, 4$ are indicated at Figure 15. (In certain cases $A, B_1, B_2, C_1, C_2, D, E$ illustrated in Figure 15, some of the domains $\Omega_j, j = 1, 4$ are absent and the value of $G^{(5)}$ is not defined there, in other

words, we skip the definition of $G^{(5)}$ for those domains). Thus, we get the new RH problem:

$$V_-^{(5)}(\xi, t, k) = V_+^{(5)}(\xi, t, k)J^{(5)}(\xi, t, k), \quad V^{(5)}(\xi, t, k) \rightarrow (1, 1), \quad k \rightarrow \infty$$

with the following jump matrices:

$$J^{(5)} = G^{(5)}|_{\Omega_1}, k \in L_1 \cup L_3; \quad J^{(5)} = (G^{(5)}|_{\Omega_2})^{-1}, k \in L_2 \cup L_4; \quad J^{(5)} = I, k \in \mathbb{R}.$$

On the interval $(i\hat{c}, -i\hat{c})$ the jump matrix is different for different situations:

–for C_1, D, E we have

$$J^{(5)} = e^{it(g_- - g_+)\sigma_3}, k \in (id_0, -id_0); \quad J^{(5)} = J^{(4)}, k \in (i\hat{c}, id_0) \cup (-id_0, -i\hat{c});$$

–for the case A, B_1, B_2, C_2 we have

$$\begin{aligned} J^{(5)}(\xi, t, k) &= \begin{pmatrix} 0 & -\frac{\delta_+ \delta_- \Lambda^2 e^{-it(g_- + g_+)}}{f(k)} \\ \frac{f(k) e^{it(g_- + g_+)}}{\delta_- \delta_+ \Lambda^2} & 0 \end{pmatrix}, k \in (i*, 0), \\ &= \begin{pmatrix} 0 & \overline{f(\bar{k})} \Lambda^2 \delta_- \delta_+ e^{-it(g_- + g_+)} \\ \frac{-e^{it(g_- + g_+)}}{\overline{f(\bar{k})} \Lambda^2 (k) \delta_- \delta_+} & 0 \end{pmatrix}, k \in (0, -i*), \\ &= J^{(4)}, k \in (i\hat{c}, id_1) \cup (-id_1, -i\hat{c}), \end{aligned}$$

where instead of $*$ we substitute d_1 in case B_1 , and substitute \hat{c} in cases A, B_2, C_2 .

These expressions for the jump matrix on the interval $(i\hat{c}, -i\hat{c})$ are calculated using (3.36)-(3.39), the jump relation

$$f(k) = r_-(k) - r_+(k), \quad k \in (0, i\hat{c})$$

and the symmetry relation $r(k) = \overline{r(-\bar{k})}$.

Step 6. Opening lenses (concerns only the cases B_1, C_1, D (see Figure 15)). Here we deal with oscillation jump matrices on the interval $i\hat{c}, -i\hat{c}$, providing triangular factorization as we did for the real line in Step 5. First we note that the function $f(k) = \frac{z(k+0)}{ka(k-0)a(k+0)}$ has the following analytic continuation from the interval $(-i\hat{c}, i\hat{c})$:

$$f(k) = \widehat{f}_+(k), \quad k \in (-i\hat{c}, i\hat{c}), \quad \text{where} \quad \widehat{f}(k) = \frac{z(k)}{ka(k)b(\bar{k})}. \quad (4.61)$$

Following ideas from [32], [33], we factorize the jump matrix $J^{(5)}$ on $(i\hat{c}, id_0) \cup (-i\hat{c}, -id_0)$ as follows:

$$J^{(5)} = F_+^{-\sigma_3} \begin{pmatrix} 1 & \frac{F_+^2 \Lambda^2 \delta_+^2 e^{-2itg_+}}{\widehat{f}_+(k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{-F_-^2 \Lambda^2 \delta_-^2 e^{-2itg_-}}{\widehat{f}_-(k)} \\ 0 & 1 \end{pmatrix} F_-^{\sigma_3} \quad (4.62)$$

for $k \in (i\hat{c}, id_0)$ and

$$J^{(5)} = F_+^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{e^{2itg_+(k, \xi)}}{\widehat{f}_+(\bar{k}) \Lambda^2 F_+^2 \delta_+^2} & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-e^{2itg_-(k, \xi)}}{\widehat{f}_-(\bar{k}) \Lambda^2 F_-^2 \delta_-^2} & 1 \end{pmatrix} F_-^{\sigma_3}, \quad (4.63)$$

for $k \in (-i\hat{c}, -id_0)$.

Direct calculations show that it is possible if $F(k, \xi)$ satisfies the relation

$$F_-(k, \xi) F_+(k, \xi) = \begin{cases} \frac{if(k)}{\Lambda^2 \delta_+ \delta_-}, & k \in (i\hat{c}, id_0), \\ \frac{i}{\overline{f(\bar{k})} \Lambda^2 \delta_+ \delta_-}, & k \in (-i\hat{c}, -id_0). \end{cases}$$

In case of B_1 we will also require these relations to be hold in the intervals $(id_1, 0)$ and $(0, -id_1)$.

We define $F(k, \xi) =$

$$\exp \left\{ \frac{w(k)}{2\pi i} \left[\int_{i\hat{c}}^{id_0} + \int_{id_1}^0 \right] \log \frac{if}{\delta^2 \Lambda^2} \frac{ds}{s-k} \frac{1}{w_+(s)} + \frac{w(k)}{2\pi i} \left[\int_0^{-id_1} + \int_{-id_0}^{-i\hat{c}} \right] \log \frac{i}{\overline{f} \delta^2 \Lambda^2} \frac{ds}{s-k} \frac{1}{w_+(s)} + \frac{w(k)}{2\pi i} \left[\int_{id_0}^{id_1} + \int_{-id_1}^{-id_0} \right] \frac{i\Delta}{s-k} \frac{ds}{w(s)} \right\}, \quad (4.64)$$

where we substitute d_1 with 0 for the cases C_1 and D . Here

$$w(k) = \sqrt{(k^2 + \hat{c}^2)(k^2 + d_0^2)(k^2 + d_1^2)} \quad \text{for } B_1, \quad \text{and}$$

$$w(k) = \sqrt{(k^2 + \hat{c}^2)(k^2 + d_0^2)} \quad \text{for } C_1, D.$$

To remove the essential singularity of $F(k, \xi)$ at infinity, we set

$$\Delta := \begin{cases} i \left(\left[\int_{i\hat{c}}^{id_0} + \int_{id_1}^0 \right] \log \frac{if}{\delta^2 \Lambda^2} \frac{s ds}{w_+(s)} \right) \left(\int_{id_0}^{id_1} \frac{s ds}{w(s)} \right)^{-1} & \text{for } B_1, \\ i \left(\int_{i\hat{c}}^{id_0} \log \frac{if}{\delta^2 \Lambda^2} \frac{ds}{w_+(s)} \right) \left(\int_{id_0}^0 \frac{ds}{w(s)} \right)^{-1} & \text{for } C_1, D. \end{cases} \quad (4.65)$$

Comment 4.1. *Studying conjugation properties of $\delta(k, \xi)F(k, \xi)$ on $\mathbb{R} \cup (i\hat{c}, -i\hat{c})$, it is easy to see that in the case A (see Figure 13) we have*

$$\delta(k)F(k) = \begin{cases} \frac{1}{a(k)\Lambda(k)} \sqrt{\frac{z(k)}{k}}, & \text{Im } z(k) > 0, \\ \frac{\overline{a(\bar{k})}}{\Lambda(k)} \sqrt{\frac{k}{z(k)}}, & \text{Im } z(k) < 0. \end{cases} \quad (4.66)$$

Function $F(k, \xi)$ has the following properties as a function of variable k :

- $F(k, \xi)$ is analytic outside the contour $[i\hat{c}, -i\hat{c}]$ (since by definition (4.64) F is a Cauchy type integral over contour $[i\hat{c}, -i\hat{c}]$);
- $F(k, \xi)$ does not vanish (since F is an exponential function);
- $F(k, \xi) \rightarrow 1$ as $k \rightarrow \infty$ (this is straightforward by expanding F as $k \rightarrow \infty$ and taking into account (4.65));
- $\frac{F_+(k, \xi)}{F_-(k, \xi)} = e^{i\Delta}$, $k \in (id_0, id_1) \cup (-id_1, -id_0)$, with $\Delta = \Delta(\xi)$ defined in (4.65) (this is due to Sokhotski-Plemelj formula applied to (4.64)).

In the case B_1 , we have also to factorize the jump matrix on $(id_0, id_1) \cup (-id_1, -id_0)$:

$$J^{(5)} = \begin{pmatrix} 1 & 0 \\ -\frac{r_+ e^{2itg_+}}{\Lambda^2 \delta_+^2} & 1 \end{pmatrix} e^{(itB + i\Delta)\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{r_- e^{2itg_-}}{\Lambda^2 \delta_-^2} & 1 \end{pmatrix}, \quad k \in (id_0, id_1), \quad (4.67a)$$

$$J^{(5)} = \begin{pmatrix} 1 & -\overline{r_+} \Lambda^2 \delta_+^2 e^{-2itg_+} \\ 0 & 1 \end{pmatrix} e^{(itB + i\Delta)\sigma_3} \begin{pmatrix} 1 & \overline{r_-} \Lambda^2 \delta_-^2 e^{2itg_-} \\ 0 & 1 \end{pmatrix}, \quad k \in (-id_1, -id_0). \quad (4.67b)$$

By using the factorizations (4.62), (4.63), we get the following RH problem:

$$V^{(6)} = V^{(5)} G^{(6)}(\xi, t, k), \quad V_-^{(6)} = V_+^{(6)} J^{(6)}(\xi, t, k), \quad V^{(6)} \rightarrow (1, 1), \quad k \rightarrow \infty,$$

where

$$\begin{aligned} G^{(6)}(\xi, t, k) &= F^{-\sigma_3}(k, \xi) \begin{pmatrix} 1 & \frac{F^2(k, \xi) \Lambda^2 \delta^2 e^{-2itg(k, \xi)}}{\widehat{f}(k)} \\ 0 & 1 \end{pmatrix}, \quad k \in \Omega_5 \bigcup \Omega_7, \\ &= F^{-\sigma_3}(k, \xi) \begin{pmatrix} 1 & 0 \\ \frac{e^{2itg(k, \xi)}}{\widehat{f}(\bar{k}) \Lambda^2 \delta^2 F^2(k, \xi)} & 1 \end{pmatrix}, \quad k \in \Omega_6 \bigcup \Omega_8, \\ &= F^{-\sigma_3}(k, \xi), \quad k \notin \Omega_5 \bigcup \Omega_6 \bigcup \Omega_7 \bigcup \Omega_8, \end{aligned}$$

and

$$\begin{aligned} J^{(6)}(\xi, t, k) &= F^{\sigma_3} G^{(6)}|_{\Omega_7}, \quad k \in L_{\text{ur}}, \quad = (G^{(6)}|_{\Omega_5})^{-1} F^{-\sigma_3}, \quad k \in L_{\text{ul}}, \\ &= F^{\sigma_3} G^{(6)}|_{\Omega_8}, \quad k \in L_{\text{dr}}, \quad = (G^{(6)}|_{\Omega_5})^{-1} F^{-\sigma_3}, \quad k \in L_{\text{dl}}, \end{aligned}$$

$$\begin{aligned} J^{(6)} &= \begin{pmatrix} 1 & 0 \\ \frac{-r(k)e^{2itg(k,\xi)}}{F^2(k,\xi)\Lambda^2\delta^2} & 1 \end{pmatrix}, \quad k \in L_1; \quad = \begin{pmatrix} 1 & \frac{\overline{r(\bar{k})}F^2(k,\xi)\Lambda^2\delta^2}{e^{2itg(k,\xi)}} \\ 0 & 1 \end{pmatrix}, \quad k \in L_2, \\ J^{(6)} &= \begin{pmatrix} 1 & \frac{k}{z(k)} \frac{a(k)\overline{b(\bar{k})}\delta_+^2(k)\Lambda^2(k)}{e^{2itg}} \\ 0 & 1 \end{pmatrix}, \quad k \in L_3; \quad = \begin{pmatrix} 1 & 0 \\ \frac{k}{z(k)} \frac{b(k)a(\bar{k})e^{2itg}}{\Lambda^2(k)\delta_-^2(k)} & 1 \end{pmatrix}, \quad k \in L_4, \\ J^{(6)} &= e^{(itB(\xi)+i\Delta(\xi))\sigma_3}, \quad k \in \pm(id_0, id_1), \quad = \begin{pmatrix} 0 & \mp i \\ \mp i & 0 \end{pmatrix}, \quad k \in (\pm i\hat{c}, \pm id_0) \cup (\pm id_1, 0). \end{aligned}$$

Exceptionally in the case B_1 we have also, in view of (4.67), jumps on the lenses $L_{2\text{ul}}, L_{2\text{ur}}, L_{2\text{dl}}, L_{2\text{dr}}$ surrounding the intervals (id_0, id_1) and $(-id_1, -id_0)$ (we denote the corresponding domains as $\Omega_{2\text{ur}}, \Omega_{2\text{ul}}, \Omega_{2\text{dr}}, \Omega_{2\text{dl}}$):

$$G^{(6)} = \begin{pmatrix} 1 & 0 \\ -\frac{re^{2itg}}{\Lambda^2\delta^2} & 1 \end{pmatrix}, \quad k \in \Omega_{2\text{ur}} \cup \Omega_{2\text{ul}}, \quad = \begin{pmatrix} 1 & \bar{r}\Lambda^2\delta^2 e^{-2itg} \\ 0 & 1 \end{pmatrix}, \quad k \in \Omega_{2\text{dr}} \cup \Omega_{2\text{dl}}.$$

$$\text{Then } J^{(6)} = \begin{cases} G^{(6)}|_{L_{2\text{ur}}}, & k \in L_{2\text{ur}}, \\ (G^{(6)})^{-1}|_{L_{2\text{ul}}}, & k \in L_{2\text{ul}}, \\ G^{(6)}|_{L_{2\text{dr}}}, & k \in L_{2\text{dr}}, \\ (G^{(6)})^{-1}|_{L_{2\text{dl}}}, & k \in L_{2\text{dl}}. \end{cases}$$

Step 7. (Hyper-) elliptic model problem.

The jump matrix $J^{(6)}$ is close to the identity matrix everywhere except for the interval $(i\hat{c}, -i\hat{c})$.

We preceded the estimation of the contribution of the contours $L_j, j = 1, 2, 5, 6, 7, 8$, which will be done in the next step, by solving of the model problem with the jump matrix supported only on the interval $(i\hat{c}, -i\hat{c})$.

Comment 4.2. *The contribution to the asymptotics of the points $\pm id_j, \pm k_j, j = 0, 1$, can be estimated by constructing the appropriate parametrices in the vicinities of these points. Parametrices in the vicinities of the points $\pm k_j, j = 0, 1$, can be constructed in terms of the parabolic cylinder functions (cf. [23]), while the parametrices in the vicinities of the points $\pm id_j, j = 0, 1$,*

can be constructed in terms of the Airy functions (cf. [4], section 4.2.4.2). However, in the case of step-like initial data, the input of the parametrices is not of the leading order, and the leading order asymptotics is already quite cumbersome. Hence, we confine ourselves to a rough estimation of the input of the points $\pm id_j$, $\pm k_j$, $j = 0, 1$, without calculation of the explicit formulas.

Hence, we introduce a model problem

$$V_-^{(mod)} = V_+^{(mod)} J^{(mod)}, \quad V^{(mod)} \rightarrow I \quad \text{as } k \rightarrow \infty, \quad \text{where}$$

$$J^{(mod)}(\xi, t, k) = \begin{cases} e^{(itB(\xi) + i\Delta(\xi))\sigma_3}, & k \in (id_0, id_1) \cup (-id_1, -id_0) \\ \begin{pmatrix} 0 & \mp i \\ \mp i & 0 \end{pmatrix}, & k \in (\pm i\hat{c}, \pm id_0) \cup (\pm id_1, 0). \end{cases} \quad (4.68)$$

Remark. In the cases of the phase functions g_t, g_1 , after the δ -transformation we get a singularity at the origin. This singularity is preserved up to the model problem. Nevertheless, going back in the train of transformations of the RH problem, which we did in Steps 1-6 of the present Section 4, and noticing that the singularity at 0 of the model problem and the one of $V^{(6)}$ are of the same order, since there are any lenses opening at the origin, we conclude that we are allowed to deal with this singular RH problem because coming back to V we get an ordinary solution.

a. Elliptic situation. To solve the model problem we distinguish the cases $d_1 = 0, d_0 > 0$ (elliptic case), $d_0 > d_1 > 0$ (hyperelliptic case of genus 2), and the trivial model problem which leads to the constant asymptotics ($d_1 = d_0 = 0$).

The solution of the vector elliptic model problem is very similar to one introduced in [25]. Consider the Riemann surface of the function

$$w^2(k) = (k^2 + \hat{c}^2)(k^2 + d_0^2),$$

with cuts along $(i\hat{c}, id_0)$ and $(-id_0, -i\hat{c})$, which is fixed by the condition $w(0) > 0$ on the first sheet. The a -cycle and b -cycle are introduced as in Figure 16a.

The basis of the holomorphic differential forms, the Abel mapping and the B-period τ are given as follows:

$$\omega = 2\pi i \left(\frac{dk}{w(k)} \right) \left(\int_a^P \frac{dk}{w(k)} \right)^{-1}, \quad A(P) = \int_{i\hat{c}}^P \omega, \quad \tau = \tau(\xi) := \int_b \omega < 0.$$

In terms of the theta function

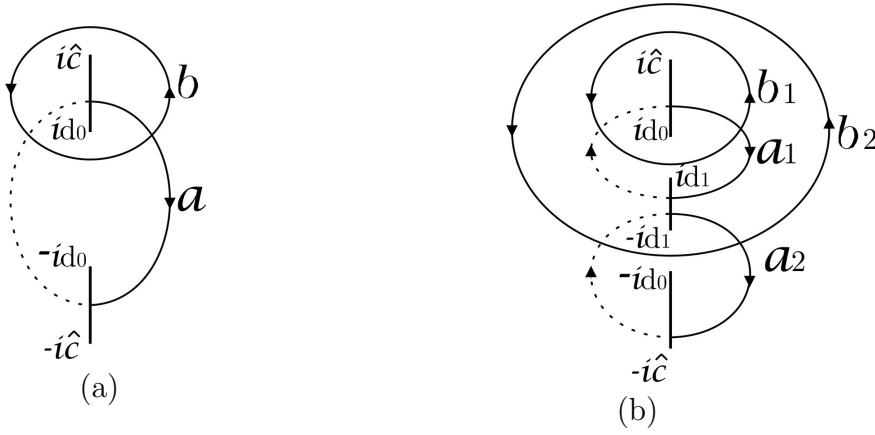


Figure 16: Riemann surfaces of genus 1 (left) and genus 2 (right).

$$\Theta(z) = \Theta(z, \tau(\xi)) = \sum_{m=-\infty}^{\infty} \exp \left\{ \frac{1}{2} \tau(\xi) m^2 + zm \right\},$$

the solution $V^{(mod)}(\xi, t; k) = \left(V_{[1]}^{(mod)}(\xi, t; k), V_{[2]}^{(mod)}(\xi, t; k) \right)$ of the model problem is given as follows:

$$V_{[1]}^{(mod)} = \sqrt[4]{\frac{k^2 + d_0^2}{k^2 + \hat{c}^2}} \frac{\Theta(A(k) - \pi i - \frac{itB + i\Delta}{2}) \Theta(A(k) - \frac{itB + i\Delta}{2}) \Theta^2(\frac{\pi i}{2})}{\Theta(A(k) - \pi i) \Theta(A(k)) \Theta(\frac{\pi i}{2} - \frac{itB + i\Delta}{2}) \Theta(\frac{\pi i}{2} + \frac{itB + i\Delta}{2})}, \quad (4.69)$$

$$V_{[2]}^{(mod)} = \sqrt[4]{\frac{k^2 + d_0^2}{k^2 + \hat{c}^2}} \frac{\Theta(-A(k) - \pi i - \frac{itB + i\Delta}{2}) \Theta(-A(k) - \frac{itB + i\Delta}{2}) \Theta^2(\frac{\pi i}{2})}{\Theta(A(k) - \pi i) \Theta(A(k)) \Theta(\frac{\pi i}{2} - \frac{itB + i\Delta}{2}) \Theta(\frac{\pi i}{2} + \frac{itB + i\Delta}{2})}. \quad (4.70)$$

Comment 4.3. Further we will need also the matrix solution $M^{(mod)}(\xi, t; k)$ of the model problem, which is fixed by asymptotics $M^{(mod)}(\xi, t; k) \rightarrow I$ as $k \rightarrow \infty$, and hence possesses the property $\det M^{(mod)} \equiv 1$. It can be constructed in the same way as in [4], section 4.2.4.2. Let us notice that the vector and matrix solutions are connected by the formula $V^{(mod)} = (1, 1)M^{(mod)}$. The model solution possesses the following behavior in the vicinities of the points $\pm id_0$:

$$M_{ij}^{(mod)} = O((k - id_0)^{-1/4}), \quad i, j = 1, 2,$$

and this estimate is uniform with respect to ξ, t . The exact form of the matrix solution is not important for us, hence we don't write it here.

b. Hyperelliptic model problem. To solve the model problem (4.68) with $d_1 > 0$ we introduce the Riemann surface X of the function

$$w^2(k) = (k^2 + \hat{c}^2)(k^2 + d_0^2)(k^2 + d_1^2)$$

as in [33]. The upper and the lower sheets of the surface are two complex

planes which are merged along the cuts $[\hat{ic}, id_0]$, $[id_1, -id_1]$ and $[-id_0, -\hat{ic}]$. The square root is fixed by the condition that on the upper sheet of this surface $w(1) > 0$. The basis $\{\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2\}$ of cycles of this Riemann surface is as follows (see Figure 16b). The \mathbf{a}_1 -cycle starts on the upper sheet from the right side of the cut $[\hat{ic}, id_0]$, goes to the right side of the cut $[id_1, -id_1]$, proceeds to the lower sheet, and then returns to the starting point. The \mathbf{b}_1 -cycle is a closed counterclockwise oriented simple loop around the cut $[\hat{ic}, id_0]$. The \mathbf{a}_2 -cycle starts on the upper sheet from the right side of the cut $[id_1, -id_1]$, goes to the right side of the cut $[-id_0, -\hat{ic}]$, proceeds to the lower sheet, and then returns to the starting point. The \mathbf{b}_2 -cycle is a closed counterclockwise oriented simple loop around the segment $[\hat{ic}, -id_1]$.

Introduce the basis $d\omega = \begin{pmatrix} d\omega_1 \\ d\omega_2 \end{pmatrix}$ of the normalized holomorphic differentials:

$$d\omega_1 = \frac{(c_1 k + c_2) dk}{w(k)}, \quad d\omega_2 = \frac{(c_1 k - c_2) dk}{w(k)}, \quad \text{where}$$

$$c_1 = \frac{\pi i}{\int_{a_1} \frac{k dk}{w(k, \xi)}} \in (-\infty, 0), \quad c_2 = \frac{\pi i}{\int_{a_1} \frac{dk}{w(k, \xi)}} \in (0, -i\infty).$$

In terms of the B -period $B = B(\xi)$

$$\left(\left(B_{jl} = \int_{b_j} d\omega_l \right) \right) = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix}, \quad B_j := \int_{b_1} d\omega_j, \quad j = 1, 2, \quad B_1 < B_2 < 0,$$

the corresponding theta function is given as follows

$$\Theta(z) = \Theta(z|B(\xi)) = \sum_{m \in \mathbb{Z}^2} \exp \left\{ \frac{1}{2} (B(\xi)m, m) + (z, m) \right\}, \quad z \in \mathbb{C}^2.$$

Since B is a symmetric matrix, we have $\Theta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \Theta \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}$. (4.71)

Introduce the Abel map $A : X \rightarrow \mathbb{C}^2 / (2\pi i \mathbb{Z}^2 + B(\xi) \mathbb{Z}^2)$, $A(P) = \int_{\hat{ic}}^P d\omega$

and the functions $\hat{\varphi}(k, \xi), \hat{\psi}(k, \xi) : \{\text{the first sheet of } X\} \rightarrow \mathbb{C}$

$$\begin{aligned} \hat{\varphi}_j(k, \xi) &= \frac{\Theta(A(k) - A(D_j) - K - \frac{1}{2}(itB_g(\xi) + i\Delta(\xi))(1, 1)^T)}{\Theta(A(k) - A(D_j) - K)}, \\ \hat{\psi}_j(k, \xi) &= \frac{\Theta(-A(k) - A(\hat{D}_j) - K - \frac{1}{2}(itB_g(\xi) + i\Delta(\xi))(1, 1)^T)}{\Theta(-A(k) - A(\hat{D}_j) - K)}, \end{aligned} \quad j = 1, 2,$$

with the vector $K = (K_1, K_2)^T$ of Riemannian constants and nonspecial divisors

$$K \equiv \left(\begin{array}{c} \frac{B_1}{2} + \frac{B_2}{2} \\ \pi i + \frac{B_1}{2} + \frac{B_2}{2} \end{array} \right), \quad \widehat{D}_1 = \{id_0 + 0, 0^+ + 0\}, \quad \widehat{D}_2 = \{-id_0 + 0, 0^+ + 0\},$$

consisting of the branch points $\pm id_0$ and the point on the upper sheet 0^+ . Since we consider a cut complex plane, it does matter at which side of the bank the point is taken, hence we write ± 0 to specify this.

Introduce also the function

$$\gamma(k) = \sqrt[4]{\frac{k^2 + d_0^2}{k^2 + \hat{c}^2} \cdot \frac{k^2}{k^2 + d_1^2}},$$

which is analytic outside of the union of segments $[i\hat{c}, id_0] \cup [id_1, -id_1] \cup [-id_0, -i\hat{c}]$.

It is easy to see that

$$A(id_0 + 0) = \begin{pmatrix} -B_1/2 \\ -B_2/2 \end{pmatrix}, \quad A(-id_0 + 0) = \begin{pmatrix} -B_2/2 + \pi i \\ -B_1/2 + \pi i \end{pmatrix} \quad (4.72a)$$

$$A(\infty) = \begin{pmatrix} i\varphi \\ i(\pi - \varphi) \end{pmatrix} \quad \text{with } \varphi = \frac{\pi}{2} + \int_{i\hat{c}}^{+i\infty} \frac{c_2 dk}{w(k)} > \frac{\pi}{2}, \quad (4.72b)$$

$$A(0^+ + 0) = \begin{pmatrix} x + \pi i \\ x \end{pmatrix} \quad \text{with } x = \frac{-B_1 - B_2}{4} + \int_{id_1 + 0}^{0^+ + 0} \frac{c_1 k dk}{w(k)} \in \mathbb{R}. \quad (4.72c)$$

Concerning asymptotics as $k \rightarrow \infty$, we notice that properties (4.71), (4.72) imply $\widehat{\varphi}_1(\infty)\widehat{\varphi}_2(\infty) = \widehat{\psi}_1(\infty)\widehat{\psi}_2(\infty)$, and hence,

$$\widehat{V}(\xi, t; k) := \left(\gamma(k) \frac{\widehat{\varphi}_1(k)\widehat{\varphi}_2(k)}{\widehat{\varphi}_1(\infty)\widehat{\varphi}_2(\infty)}, \quad \gamma(k) \frac{\widehat{\psi}_1(k)\widehat{\psi}_2(k)}{\widehat{\psi}_1(\infty)\widehat{\psi}_2(\infty)} \right) \quad (4.73)$$

solves the hyperelliptic model problem.

Comment 4.4. *Further we will need also the matrix solution $M^{(mod)}(\xi, t; k)$ of the model problem, which is fixed by asymptotics $M^{(mod)}(\xi, t; k) \rightarrow I$ as $k \rightarrow \infty$, and hence possesses the property $\det M^{(mod)} \equiv 1$. It can be constructed in the same way as in [4], section 4.2.4.2. Let us notice that the vector and*

matrix solutions are connected by the formula $V^{(mod)} = (1, 1)M^{(mod)}$. The model solution possesses the following behavior in the vicinities of the points $\pm id_l$, $l = 1, 2$:

$$M_{ij}^{(mod)} = O((k - id_l)^{-1/4}), \quad l, i, j = 1, 2,$$

and this estimate is uniform with respect to ξ, t . The exact form of the matrix solution is not important for us, hence we don't write it here.

c. Constant model problem. Finally, in the case when $d_0 = d_1 = 0$ (see cases A, B_2, C_2 of Figure 15), which is a trivial case, the solution of the model problem (4.68) is given by

$$V^{(mod)}(\xi, t; k) = \left(\sqrt[4]{\frac{k^2}{k^2 + \hat{c}^2}}, \sqrt[4]{\frac{k^2}{k^2 + \hat{c}^2}} \right).$$

Step 8. Estimate of the contribution of the jumps on contours L_j , $j = 1, \dots, 8$ and final asymptotics.

We start with the following lemma:

Lemma 4.1. *Let $V^{(6)}(\xi, t; k)$ be the (vector-valued) solution of the RH problem from Step 6 of Section 4 (with jump contour as in Figure 15); $V^{(mod)}(\xi, t; k)$ and $M^{(mod)}(\xi, t; k)$ be the vector and matrix-valued solutions of the corresponding model problem from Step 7 (with jump matrix $[\hat{ic}, -\hat{ic}]$). Define the correction function $V_{err} = V^{(6)}(M^{(mod)})^{-1}$. Denote*

$$V_{err}(\xi, t; k) =: V_{err}^{(0)}(\xi, t) + V_{err}^{(1)}(\xi, t)(k - i/2) + V_{err}^{(2)}(\xi, t; k)(k - i/2)^2.$$

Then

$$|V_{err,j}^{(0)}(\xi, t) - 1| = O(t^{-1/2}), \quad |V_{err,j}^{(l)}(\xi, t; k)| = O(t^{-1/2}), \quad j = 1, 2, \quad l = 1, 2,$$

uniformly for $k \in \mathbb{C}$, $\xi_m + \varepsilon < \xi < \xi_{m+1} - \varepsilon$ as $t \rightarrow \infty$, $m = 1, 2, 3$, where $\varepsilon > 0$ is a sufficiently small positive number.

Sketch of the proof.

For definiteness let us consider the case C_1 , when both types of the points $\pm id_0$ and $\pm k_0$ are presented. The other cases can be treated similarly.

The jump contour $\Sigma_{err} := \Sigma^{(6)} \setminus [\hat{ic}, -\hat{ic}]$ for the correction function V_{err} contains the following points of nonuniformity of the jump matrix (see Figures 15, 13):

- the real points $\pm k_0 \in \mathbb{R}$;

- the imaginary points $\pm i\mu_0 \in [i\hat{c}, -i\hat{c}]$;
- the end points $\pm i\hat{c}$.

The jump matrix $J^{(6)}$ is not close to the identity matrix in L_∞ sense. However, the jump matrices in the vicinities of the points $\pm k_0$, $\pm id_0$ are highly oscillatory, but uniformly bounded. This provides that the solution of the RH problem is also uniformly bounded,

$$|V_i^{(6)}| \leq C, \quad |M_{ij}^{(6)}| \leq C, \quad i, j = 1, 2,$$

where C is a constant that does not depend on ξ, t, k .

The jump matrix J_{err} for the correction function V_{err} is equal to

$$J_{err} = M_+^{(mod)} J^{(6)} (J^{(mod)})^{-1} (M_+^{(mod)})^{-1} = \begin{cases} I, & k \in (i\hat{c}, -i\hat{c}), \\ M^{(mod)} J^{(6)} (M^{(mod)})^{-1}, & k \in \Sigma^{(6)} \setminus [i\hat{c}, -i\hat{c}], \end{cases}$$

where, as usual, by $+/-$ we denote the limit from the positive/ negative side of the contour. By Sokhotski-Plemelj formula, we have the standard representation of V_{err} :

$$V_{err}(k) = (1, 1) + \frac{1}{2\pi i} \int_{\Sigma_{err}} \frac{V_{err,+}(s)(I - J_{err})ds}{s - k},$$

and hence (we denote here $E := (1, 1)$),

$$\begin{aligned} V_{err}(k) &= E + \int_{\Sigma_{err}} \frac{F(s)ds}{2\pi i(s - \frac{i}{2})} + \int_{\Sigma_{err}} \frac{F(s)ds (k - \frac{i}{2})}{2\pi i(s - \frac{i}{2})^2} + \int_{\Sigma_{err}} \frac{F(s)ds (k - \frac{i}{2})^2}{2\pi i(s - \frac{i}{2})^2(s - k)} = \\ &= V_{err}^{(0)} + V_{err}^{(1)}(k - i/2) + V_{err}^{(2)}(k - i/2). \end{aligned}$$

where $F(\xi, t; s) := V_+^{(6)}(\xi, t; s)M_{mod}^{-1}(\xi, t; s)(I - J_{err}(\xi, t; s))$. Let estimate, for example, $V_{err}^{(1)}$. Quantities $V_{err}^{(0)} - (1, 1)$ and $V_{err}^{(2)}$ can be estimated similarly.

Let us denote the parts of the contour Σ_{err} in the vicinities of the points $\pm id_0$, $\pm k_0$, as follows:

- $[id_0]$: $L_{1,\pm}$, depending on whether $\operatorname{Re} k > 0$ or $\operatorname{Re} k < 0$;
- $[-id_0]$: $L_{2,\pm}$, depending on whether $\operatorname{Re} k > 0$ or $\operatorname{Re} k < 0$;

- $[ik_0] : L_{j,\mathfrak{r}}, j = 1, 2, 3, 4;$
- $[-ik_0] : L_{j,\mathfrak{l}}, j = 1, 2, 3, 4.$

The rest of the contour Σ_{err} we denote by Σ_{err}^C . Let us notice, that the points $\pm i\hat{c}$ can be excluded from analysis. Indeed, by moving up slightly the lenses $L_{ul}, L_{u,\mathfrak{r}}$ towards a point $i(\hat{c} + \delta)$, $\delta > 0$, we transform the oscillatory behavior of jump matrices into exponentially decaying. The same is for the point $-i\hat{c}$.

1. To estimate $F(s)$ on $L_{1,\pm}$, let us expand the function $g(k, \xi)$ in a vicinity of the point id_0 :

$$g(k) = g_{\pm} + g^{(3/2)} \left(\frac{k - id_0}{-i} \right)^{3/2} (1 + O(k - id_0)), k \rightarrow id_0 \pm 0,$$

with

$$g_{\pm} \in \mathbb{R}, \quad g^{(3/2)} > 0,$$

where the sign $+$ or $-$ is taken for $L_{1,\pm}$, respectively. Let us make a change of variable on $L_{1,+}$:

$$g(k) = g_{\pm} + g^{(3/2)} (e^{i\alpha} u)^{3/2}, \quad \frac{k - id_0}{-i} = e^{i\alpha} u (1 + O(u)), \quad u \rightarrow 0, u > 0,$$

where $0 < \alpha < \pi/3$ is the angle which $L_{1,+}$ makes with the ray $(id_0, 0)$. Let us notice, that we can choose $L_{1,\mathfrak{r}}$ in such a way that $u > 0$ is real. Hence, we can estimate $F(s)$ on $L_{1,\pm}$ as

$$|F_j(s)| \leq \frac{C}{\sqrt[4]{|k - id_0|}} \left| e^{-2tg^{(3/2)} \sin \frac{3\alpha}{2} u^{3/2}} \right|, \quad j = 1, 2,$$

where $C > 0$ is a generic constant that does not depend on ξ, t, k . Hence,

$$\int_{L_{1,+}} \frac{F(s) ds}{(s - \frac{i}{2})^2} \leq \frac{C}{\text{dist}^2(\Sigma_{err}, i/2)} \int_0^{+\infty} \frac{1}{\sqrt[4]{u}} e^{-2tg^{(3/2)} \sin \frac{3\alpha}{2} u^{3/2}} du \leq Ct^{-1/2}.$$

Integrals over the contours $L_{1,-}, L_{2,\pm}$ can be estimated similarly.

2. To estimate $F(s)$ on $L_{3,\mathfrak{r}}$, let us expand the function $g(k, \xi)$ in a vicinity of the point k_0 :

$$g(k) = g^{(0)} + g^{(2)}(k - k_0)^2(1 + O(k - k_0)),$$

with

$$g^{(0)} \in \mathbb{R}, \quad g^{(2)} < 0.$$

Let us make a change of variable on $L_{1,+}$:

$$g(k) = g^{(0)} + g^{(2)} (e^{i\beta} v)^2, \quad k - ik_0 = e^{i\beta} v(1 + O(v)), \quad v \rightarrow 0, v > 0,$$

where $0 < \beta < \pi/2$ is the angle which $L_{3,\mathfrak{r}}$ makes with the ray $(ik_0, +\infty)$. Let us notice, that we can choose $L_{3,\mathfrak{r}}$ in such a way that $v > 0$ is real. Hence, we can estimate $F(s)$ on $L_{3,\mathfrak{r}}$ as

$$|F_j(s)| \leq \left| e^{2tg^{(2)} \sin 2\beta v^2} \right|, \quad j = 1, 2,$$

where $C > 0$ is a generic constant that does not depend on ξ, t, k . Hence,

$$\int_{L_{3,\mathfrak{r}}} \frac{F(s)ds}{\left(s - \frac{i}{2}\right)^2} \leq \frac{C}{\text{dist}^2(\Sigma_{err}, i/2)} \int_0^{+\infty} e^{2tg^{(2)} \sin 2\beta v^2} dv \leq Ct^{-1/2}.$$

Integrals over the contours $L_{1,-}$, $L_{2,\pm}$ can be estimated similarly.

Since the integral over Σ_{err}^C is exponentially small, we have that

$$V_{err,j}^{(1)} = \int_{\Sigma_{err}} \frac{F_j(s)ds}{2\pi i \left(s - \frac{i}{2}\right)^2} = O(t^{-1/2}), \quad j = 1, 2.$$

□

Comment 4.5. *The estimate of the integrals in the vicinities of the points $\pm k_0, \pm k_1$ is sharp, while the estimate of integrals in the vicinities of the points $\pm i\mu_0, \pm i\mu_1$ is a rough one: the actual behavior of $V_{err} - (1, 1)$ is of the order t^{-1} .*

Lemma 4.2. *Let $u^{(mod)}(x, t)$, $m^{(mod)}(x, t)$ be defined as in (2.31a, b), where we substitute V instead of $V^{(mod)}$. Then for sufficiently small $\varepsilon > 0$,*

$$u(x, t) = u^{(mod)}(x - \tilde{x}(x, t), t)(1 + O(t^{-1/2})), \quad m(x, t) = m^{(mod)}(x, t)(1 + O(t^{-1/2})),$$

uniformly for $(\zeta_l + \varepsilon)\omega t \leq x \leq (\zeta_{l+1} - \varepsilon)\omega t$, $l = 1, 2, 3$, where

$$\tilde{x}(x, t) = \log \left[F^2\left(\frac{i}{2}\right) \delta^2\left(\frac{i}{2}, \xi\right) \Omega^2\left(\frac{i}{2}\right) e^{-2it(g-g_r)(i/2)} \right]$$

is a slowly changing function. Here $g, g_r, F, \delta, \Omega$ are defined in Section 4, and the parameter $\xi = \frac{y}{\omega t}$ is the same as in parametric definition of $u^{(mod)}$, $m^{(mod)}$.

Proof Following the chain of transformations of the RH problem made in Steps 1–6 of the present Section 4, we conclude that in a neighborhood of the point $k = i/2$

$$\begin{aligned} V &= V^{(1)} e^{-it(g-g_r)\sigma_3} = V^{(2)} e^{-it(g-g_r)\sigma_3} = \text{because} = V^{(3)} \Omega^{\sigma_3}(k) e^{-it(g-g_r)\sigma_3} = \\ &= V^{(4)} \delta^{\sigma_3}(k, \xi) \Omega^{\sigma_3}(k) e^{-it(g-g_r)\sigma_3} = V^{(5)}(\xi, t; k) \delta^{\sigma_3}(k, \xi) \Omega^{\sigma_3}(k) e^{-it(g-g_r)\sigma_3} = \\ &= V^{(6)} F^{\sigma_3}(k, \xi) \delta^{\sigma_3}(k, \xi) \Omega^{\sigma_3}(k) e^{-it(g-g_r)\sigma_3}, \quad t \rightarrow \infty. \end{aligned}$$

Taking into account Lemma 4.1 and the property $V^{(mod)} = (1, 1)M^{(mod)}$, we see, that $V^{(6)} = V^{err} (M^{(mod)})^{-1}$, and since all the elements of the vector and matrix functions $V^{(mod)}$, $M^{(mod)}$ are separated from 0, we can write

$$V^{(6)} = V^{err} M^{(mod)} = V^{(mod)} \left(I + O\left(t^{-\frac{1}{2}}\right) \right).$$

Hence, from (2.31a), (2.31b), we obtain

$$e^{x-y} = \frac{V_{[1]}^{(mod)}}{V_{[2]}^{(mod)}} \bigg|_{k=\frac{i}{2}} F^2\left(\frac{i}{2}\right) \delta^2\left(\frac{i}{2}, \xi\right) \Omega^2\left(\frac{i}{2}\right) e^{-2it(g-g_r)(i/2)} \left(1 + O\left(t^{-\frac{1}{2}}\right)\right), \quad (4.74a)$$

$$\sqrt{\frac{m+\omega}{\omega}} \left(1 + \frac{2i}{\omega} u(x, t) \left(k - \frac{i}{2}\right) + O\left(k - \frac{i}{2}\right)^2\right) = V_{[1]}^{(mod)}(x, t; k) V_{[2]}^{(mod)}(x, t; k) \left(1 + O\left(t^{-\frac{1}{2}}\right)\right). \quad (4.74b)$$

The corresponding borders between the different asymptotic sectors in the x, t -half-plane are expressed in terms of the corresponding borders in the y, t -half-plane as follows:

$$\zeta \equiv \frac{x}{\omega t} = \xi - \frac{2i}{\omega} (g - g_r) \left(\frac{i}{2}, \xi\right) + O(t^{-1}), \quad t \rightarrow \infty, \quad (4.75)$$

■

Finally, Lemma 4.2 and formulas (4.69), (4.70), (4.73) with straight, but cumbersome calculations lead to

Theorem 4.1. Elliptic asymptotics (genus 1). Under Assumptions 1-4 of Section 1, the asymptotics of the solution of the Cauchy problem (1.1)-(1.2) can be described as follows. Let $\varepsilon > 0$ be a sufficiently small positive number. Let $\zeta_j, j = 2, 3, 4$, be as in (3.40)-(3.41).

Then for $t \rightarrow \infty$, in both of the following two cases (see Figure 13):

1. (case C_1) $\frac{c}{\omega} > 3$ or $1 < \frac{c}{\omega} < 3$, and $(\zeta_2 + \varepsilon)\omega t < x < (\zeta_3 - \varepsilon)\omega t$,
2. (case D) $\frac{c}{\omega} > 3$, $1 < \frac{c}{\omega} < 3$, or $0 < \frac{c}{\omega} < 1$, and $\zeta \equiv \frac{x}{\omega t} \in (\zeta_3 + \varepsilon)\omega t < x < (\zeta_4 - \varepsilon)\omega t$,

the asymptotics of the solution of the Cauchy problem (1.1)-(1.2) is given parametrically by the following formulas:

$$x = y - 2it(g - g_{\tau})(i/2) + \log F^2(i/2, \xi) + \log \delta^2(i/2, \xi) + \log \Lambda^2(i/2) + E\left(\frac{tB(\xi) + \Delta(\xi)}{2}\right) + O(t^{-\frac{1}{2}}), \quad (4.76)$$

$$u(x, t) = \frac{c(1 - d_0^2(\xi)\hat{c}^{-2})}{(1 - 4d_0^2(\xi))} + \Gamma(\xi) \left(E' \left(\frac{tB(\xi) + \Delta(\xi)}{2} \right) - E'(0) \right) + O(t^{-\frac{1}{2}}), \quad \xi = \frac{y}{t}, \quad (4.77)$$

$$\sqrt{\frac{m(x, t) + \omega}{\omega}} = \sqrt{\frac{1 - 4d_0^2}{1 - 4\hat{c}^2} \frac{G\left(\pi + \frac{tB + \Delta}{2}\right) G\left(\frac{tB + \Delta}{2}\right)}{G(\pi) G(0)} \frac{H^2(0)}{H^2\left(\frac{tB + \Delta}{2}\right)}}$$

with $g \equiv g_2$ in the first case (C_1) and $g \equiv g_3$ in the second case (D). Here,

$$E(U) = \log \frac{\Theta(iU - A(i/2) + \pi i) \Theta(iU - A(i/2))}{\Theta(iU + A(i/2) + \pi i) \Theta(iU + A(i/2))},$$

$$G(U) = \Theta\left(A\left(\frac{i}{2}\right) + iU\right) \Theta\left(A\left(\frac{i}{2}\right) - iU\right), \quad H(u) = \Theta\left(\frac{\pi i}{2} + iU\right) \Theta\left(\frac{\pi i}{2} - iU\right),$$

$$\Gamma(\xi) = \frac{\pi\omega}{4w\left(\frac{i}{2}\right) \int_0^{d_0} \frac{ds}{\sqrt{(\hat{c}^2 - s^2)(d_0^2 - s^2)}}}, \quad \hat{c} = \sqrt{\frac{c}{4(c + \omega)}}.$$

Theorem 4.2. Hyperelliptic asymptotics (genus 2). Under Assumptions 1-4 of Section 1 the asymptotics of the solution of the Cauchy problem (1.1)-(1.2) can be described as follows.

Let $\frac{c}{\omega} > 3$ and $\varepsilon > 0$ be a sufficiently small positive number. Let ζ_2 be as in

(3.40)–(3.41). Then for $t \rightarrow \infty$ and $(\zeta_1 + \varepsilon)\omega t < x < (\zeta_2 - \varepsilon)\omega t$ (case B_1 in Figure 13), the asymptotics of the solution of the Cauchy problem (1.1)–(1.2) is given parametrically in terms of the modulated hyperelliptic functions:

$$\begin{aligned} x = & y - 2it(g - g_{\mathfrak{r}})\left(\frac{i}{2}\right) + \log F^2\left(\frac{i}{2}, \xi\right) + \log \delta^2\left(\frac{i}{2}, \xi\right) + \log \Lambda^2\left(\frac{i}{2}\right) \\ & + E\left(\frac{tB(\xi) + \Delta(\xi)}{2}\right) + o(1), \end{aligned} \quad (4.78)$$

$$\begin{aligned} \frac{2iu}{\omega} = & \frac{8i(\hat{c}^2 - d_0^2 + d_1^2 - 8\hat{c}^2 d_1^2 + 16\hat{c}^2 d_0^2 d_1^2)}{(1 - 4\hat{c}^2)(1 - 4d_0^2)(1 - 4d_1^2)} + \\ & + H\left(A(i/2), A(\hat{D}_1), \frac{tB + \Delta}{2}\right) - H\left(-A(i/2), A(\hat{D}_1), \frac{tB + \Delta}{2}\right) + \\ & + H\left(A(i/2), A(\hat{D}_2), \frac{tB + \Delta}{2}\right) - H\left(-A(i/2), A(\hat{D}_2), \frac{tB + \Delta}{2}\right) \\ & - H\left(A(i/2), A(\hat{D}_1), 0\right) + H\left(-A(i/2), A(\hat{D}_1), 0\right) \\ & - H\left(A(i/2), A(\hat{D}_2), 0\right) + H\left(-A(i/2), A(\hat{D}_2), 0\right), \end{aligned} \quad (4.79)$$

$$\sqrt{\frac{m + \omega}{\omega}} = \sqrt{\frac{1 - 4d_0^2}{1 - 4\hat{c}^2} \cdot \frac{1}{1 - 4d_1^2} \cdot \frac{G\left(\frac{tB + \Delta}{2}\right)}{G(0)} \frac{H^2(0)}{H^2\left(\frac{tB + \Delta}{2}\right)}}.$$

Here,

$$\begin{aligned} E(U) = & \log \left[\frac{\Theta\left(iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} - A(i/2) + A(\hat{D}_1) + K\right)}{\Theta\left(iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A(i/2) + A(\hat{D}_1) + K\right)} \right] \times \\ & \times \frac{\Theta\left(iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} - A(i/2) + A(\hat{D}_2) + K\right)}{\Theta\left(iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A(i/2) + A(\hat{D}_2) + K\right)} \times \end{aligned}$$

$$\begin{aligned}
& \times \frac{\Theta \left(A(i/2) + A(\widehat{D}_1) + K \right)}{\Theta \left(-A(i/2) + A(\widehat{D}_1) + K \right)} \times \frac{\Theta \left(A(i/2) + A(\widehat{D}_2) + K \right)}{\Theta \left(-A(i/2) + A(\widehat{D}_2) + K \right)} \Big] \\
H(U_1, U_2, U_3) &= \frac{\partial_1 \Theta \left(U_1 - U_2 - K - iU_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \frac{c_1 i/2 + c_2}{w(i/2)}}{\Theta \left(U_1 - U_2 - K - iU_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)} + \\
&+ \frac{\partial_2 \Theta \left(U_1 - U_2 - K - iU_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \frac{c_1 i/2 - c_2}{w(i/2)}}{\Theta \left(U_1 - U_2 - K - iU_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)} \quad (4.80)
\end{aligned}$$

and

$$\begin{aligned}
G(U) &= \Theta \left(A\left(\frac{i}{2}\right) - A(\widehat{D}_1) - K - iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \Theta \left(-A\left(\frac{i}{2}\right) - A(\widehat{D}_1) - K - iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\
&\times \Theta \left(A\left(\frac{i}{2}\right) - A(\widehat{D}_2) - K - iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \Theta \left(-A\left(\frac{i}{2}\right) - A(\widehat{D}_2) - K - iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \\
H(U) &= \Theta \left(A(\infty) - A(\widehat{D}_1) - K - iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \Theta \left(A(\infty) - A(\widehat{D}_2) - K - iU \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right).
\end{aligned}$$

Using the formulas (4.74), (4.74) and (4.75), we get

Theorem 4.3. *Under Assumptions 1-4 of Section 1 the asymptotics of the solution of the Cauchy problem (1.1)-(1.2) can be described as follows.*

Let $\varepsilon > 0$ be a sufficiently small positive number.

Then for $t \rightarrow \infty$, in all of the following three cases (see Figure 13):

1. (case A) $\frac{c}{\omega} \in (0, 1) \cup (1, 3) \cup (3, \infty)$ and $-\infty < x < \left(\frac{3c-\omega}{4\omega} - \varepsilon\right) \omega t$,
2. (case B₂) $\frac{c}{\omega} \in (1, 3)$ and $\left(\frac{3c-\omega}{4\omega} + \varepsilon\right) \omega t < x < \left(\frac{2\omega^2 - c\omega + c^2}{\omega(c+\omega)} - \varepsilon\right) \omega t$,
or
 $\frac{c}{\omega} \in (0, 1)$ and $\left(\frac{3c-\omega}{4\omega} + \varepsilon\right) \omega t < x < \left(\frac{c}{\omega} - \varepsilon\right) \omega t$,
3. (case C₂) $\frac{c}{\omega} \in (0, 1)$ and $\left(\frac{c}{\omega} + \varepsilon\right) \omega t < x < \left(\frac{2\omega^2 - c\omega + c^2}{\omega(c+\omega)} - \varepsilon\right) \omega t$,

the main term of asymptotics of the solution of the initial value problem (1.1), (1.2), (1.8), (1.6) is equal to the initial constant:

$$u(x, t) = c + O(t^{-1/2}).$$

$$m(x, t) = c + O(t^{-1/2}).$$

In the 1st case (A), the estimate can be refined:

$$u(x, t) = c + O(e^{-Ct}).$$

$$m(x, t) = c + O(e^{-Ct}),$$

where $C = C(\varepsilon) > 0$ is a positive number.

Proof. Indeed, the corresponding borders in the x, t -half-plane $\zeta = \frac{x}{\omega t}$ in terms of the borders in the y, t -half-plane $\xi = \frac{y}{\omega t}$ are calculated via (4.75):

1. Let $\frac{c}{\omega} \in (0, 1) \cup (1, 3) \cup (3, \infty)$. Then $\xi_1 = -\frac{1}{4} \left(\frac{c+\omega}{\omega} \right)^{3/2}$ and $\zeta_1 = \frac{3c-\omega}{4\omega}$.
2. Let $\frac{c}{\omega} \in (1, 3)$. Then $\xi_2 = \frac{-2(c-\omega)}{\sqrt{\omega(c+\omega)}}$ and $\zeta_2 = \frac{2\omega^2 - c\omega + c^2}{\omega(c+\omega)}$.
3. Let $\frac{c}{\omega} \in (0, 1)$. Then $\xi_2 = 0$, $\xi_3 = \frac{-2(c-\omega)}{\sqrt{\omega(c+\omega)}}$ and $\zeta_2 = \frac{c}{\omega}$, $\zeta_3 = \frac{2\omega^2 - c\omega + c^2}{\omega(c+\omega)}$.

□

5. Asymptotics in the domains $\frac{x}{\omega t} > 2 \left(\frac{c}{\omega} + 1 \right)$ and $\frac{x}{\omega t} < \frac{3c-\omega}{4\omega}$.

For the full description of asymptotic behavior of solution to the Cauchy problem (1.1), (1.2), we also include the following theorem, which describes the asymptotics in the soliton region. These asymptotics have been studied in ([35]). The short sketch for the solitonic region is as follows: we follow all Steps 1-4 in RH problem transformations Section 5 with initial function g_t with the following modifications for the partial transmission coefficient $\Lambda(k)$:

$$\Lambda(k, \xi) = \prod_{\kappa_0(\xi) + \sigma < \kappa_j} \frac{k + i\kappa_j}{k - i\kappa_j}, \quad \xi \geq 2 \left(\frac{c}{\omega} + 1 \right), \quad \text{where } \kappa_0 := \frac{1}{2} \sqrt{1 - 2/\xi}.$$

Here $\sigma > 0$ is a sufficiently small number. Skipping the δ -transformation of Step 4, Step 5 leads to a problem with jump matrix close to the identity matrix everywhere except for the parts of the contour close to poles $i\kappa_j$ of the transmission coefficient. Turning back to the explicitly solvable meromorphic problem with two poles and identity jump matrix, we come to

Theorem 5.1. *Under Assumptions 1-4 of Section 1, the asymptotics of the solution of the Cauchy problem (1.1)-(1.2) can be described as follows.*

Let $v_j = \frac{2}{1 - 4\kappa_j^2}$, $j = 1, \dots, N$ and let $\varepsilon > 0$ be sufficiently small such that $|\kappa_j - \kappa_l| > 4\varepsilon$ for any $j \neq l$. Then in the soliton region $x \geq (2(\frac{c}{\omega} + 1) + \delta)\omega t$ one has as $t \rightarrow \infty$

1. if $\left| \frac{x}{\omega t} - v_j \right| < \varepsilon$ for some j , then

$$u(x, t) = u_{\kappa_j, \delta, v_j, x_j}^{(sol)}(x, t) + O(e^{-Ct}), \quad m(x, t) = m_{\kappa_j, \delta, v_j, x_j}^{(sol)}(x, t) + O(e^{-Ct}),$$

where $u_j^{(sol)}(x, t)$, $m_j^{(sol)}(x, t)$ are given below by (5.81) - (5.83) and $C > 0$ is a positive constant;

2. if $\left| \frac{x}{\omega t} - \frac{2}{1 - 4\kappa_j^2} \right| \geq \varepsilon$ for all j , then

$$u(x, t) = O(e^{-Ct}), \quad m(x, t) = O(e^{-Ct}).$$

Here $C = C(\varepsilon)$ is a positive constant and $u_{\kappa, \delta, v, x_j}$, $m_{\kappa, \delta, v, x_j}$ are given parametrically by the formulas

$$x = y + \log \frac{1 + \delta e^{-2\kappa(y - \omega vt)} \frac{1+2\kappa}{1-2\kappa}}{1 + \delta e^{-2\kappa(y - \omega vt)} \frac{1-2\kappa}{1+2\kappa}} + x_j \quad (5.81)$$

$$\frac{m_{\kappa, \delta, v, x_j}^{(sol)}(x, t) + \omega}{\omega} = \left(1 + \frac{16\kappa^2}{1 - 4\kappa^2} \frac{\delta e^{-2\kappa(y - \omega vt)}}{(1 + \delta e^{-2\kappa(y - \omega vt)})^2} \right)^2, \quad (5.82)$$

$$\frac{u_{\kappa, \delta, v, x_j}^{(sol)}(x, t)}{\omega} = \frac{\frac{32\kappa^2}{(1 - 4\kappa^2)^2} \delta e^{-2\kappa(y - \omega vt)}}{(1 + \delta e^{-2\kappa(y - \omega vt)})^2 + \frac{16\kappa^2}{1 - 4\kappa^2} \delta e^{-2\kappa(y - \omega vt)}}, \quad (5.83)$$

with

$$x_j = 2 \sum_{\kappa_l \geq \kappa_0 + \varepsilon} \log \frac{1 + 2\kappa_l}{1 - 2\kappa_l}, \quad \delta = \frac{\gamma^2}{2\kappa_j \Lambda^2(i\kappa_j, \xi)}, \quad \kappa = \kappa_j, \quad v_j = \frac{2}{1 - 4\kappa_j^2}.$$

Acknowledgements. We thank Iryna Egorova, Dmitry Shepelsky, Robert Buckingham and Svetlana Roudenko for useful discussions.

The research has been supported by the project "Support of inter-sectoral mobility and quality enhancement of research teams at Czech Technical University in Prague", CZ.1.07/2.3.00/30.0034, sponsored by European Social Fund in the Czech Republic."

References

- [1] Bikbaev R F and Novokshenov V Yu 1989 Existence and uniqueness of the solution of the Whitham equation (Russian) *Asymptotic methods for solving problems in mathematical physics Akad. Nauk SSSR Ural. Otdel. Bashkir. Nauchn. Tsentr Ufa* 81-95
- [2] Bikbaev R F 1989 Structure of a shock wave in the theory of the Korteweg-de Vries equation. *Phys. Lett. A* **141/5-6** 289-293
- [3] Bikbaev R F and Sharipov R A 1989 The asymptotic behavior, as $t \rightarrow \infty$, of the solution of the Cauchy problem for the Korteweg-de Vries equation in a class of potentials with finite-gap behavior as $x \rightarrow \pm\infty$. *Teoret. Mat. Fiz.* **78/3** 345-356 translation in *Theoret. and Math. Phys.* **78/3** 244-252
- [4] Bleher, Pavel M. Lectures on random matrix models: the Riemann-Hilbert approach. Random matrices, random processes and integrable systems, 251–349, CRM Ser. Math. Phys., Springer, New York, 2011.
- [5] Thomas Bothner, Percy Deift, Alexander Its, Igor Krasovsky. On the asymptotic behavior of a log gas in the bulk scaling limit in the presence of a varying external potential I. 2015, arXiv:1407.2910.
- [6] Boutet de Monvel A, Its A R and Kotlyarov V P 2007 Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition. *C. R. Math. Acad. Sci. Paris.* **345/11** 615-620
- [7] Boutet de Monvel, Anne; Shepelsky, Dmitry. Riemann-Hilbert problem in the inverse scattering for the Camassa-Holm equation on the line. Probability, geometry and integrable systems, 53–75, Math. Sci. Res. Inst. Publ., 55, Cambridge Univ. Press, Cambridge, 2007.
- [8] Boutet de Monvel, Anne; Shepelsky, Dmitry. The Camassa-Holm equation on the half-line: a Riemann-Hilbert approach. *J. Geom. Anal.* 18 (2008), no. 2, 285–323.
- [9] Boutet de Monvel, Anne; Shepelsky, Dmitry. Long-time asymptotics of the Camassa-Holm equation on the line. Integrable systems and random matrices, 99–116, Contemp. Math., 458, Amer. Math. Soc., Providence, RI, 2008.

- [10] Boutet de Monvel, Anne; Kostenko, Aleksey; Shepelsky, Dmitry; Teschl, Gerald Long-time asymptotics for the Camassa-Holm equation. *SIAM J. Math. Anal.* 41 (2009), no. 4, 1559–1588.
- [11] Boutet de Monvel, Anne; Its, Alexander; Shepelsky, Dmitry. Painleve-type asymptotics for the Camassa-Holm equation. *SIAM J. Math. Anal.* 42 (2010), no. 4, 1854–1873.
- [12] Buckingham, Robert J.; Miller, Peter D. Large-degree asymptotics of rational Painlevé-II functions: noncritical behaviour. *Nonlinearity* 27 (2014), no. 10, 2489–2578.
- [13] Buckingham, Robert; Tovbis, Alexander; Venakides, Stephanos; Zhou, Xin The semiclassical focusing nonlinear Schrödinger equation. Recent advances in nonlinear partial differential equations and applications, 47–80, *Proc. Sympos. Appl. Math.*, 65, Amer. Math. Soc., Providence, RI, 2007.
- [14] Buckingham R and Venakides S 2007 Long-time asymptotics of the nonlinear Schrodinger equation shock problem. *Comm. Pure Appl. Math.* **60/9** 1349-1414
- [15] Buslaev, V.; Fomin, V. An inverse scattering problem for the one-dimensional Schrödinger equation on the entire axis. (Russian) *Vestnik Leningrad. Univ.* 17 (1962), no. 1, 56–64.
- [16] Camassa, R.; Holm, D. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* 71 (1993), 1661-1664.
- [17] Camassa, R.; Holm, D.; Hyman, J. A new integrable shallow water equation. *Adv. Appl. Mech.* 31 (1994), 1-33.
- [18] Claeys, T.; Vanlessen, M. Universality of a double scaling limit near singular edge points in random matrix models. *Comm. Math. Phys.* 273 (2007), no. 2, 499–532.
- [19] Cohen, Amy; Kappeler, Thomas. Scattering and inverse scattering for steplike potentials in the Schrodinger equation. *Indiana Univ. Math. J.* 34 (1985), no. 1, 127–180.
- [20] Constantin, Adrian On the scattering problem for the Camassa-Holm equation. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 457 (2001), no. 2008, 953–970.

- [21] Constantin, Adrian; Gerdjikov, Vladimir S.; Ivanov, Rossen I. Inverse scattering transform for the Camassa-Holm equation. *Inverse Problems* 22 (2006), no. 6, 2197–2207.
- [22] Dai, H.-H. Model equations for nonlinear dispersive waves in a compressible Mooney – Rivlin rod. *Acta Mechanica* 127 (1998), 293–308.
- [23] Deift, P., Zhou, X., A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. of Math.* (2), 137 (1993), no.2, 295–368
- [24] Egorova I, Gladka Z, Kotlyarov V and Teschl G 2012 Long-Time Asymptotics for the Korteweg-de Vries Equation with Steplike Initial Data. *Nonlinearity* **26/7** 1839–1864
- [25] Egorova, Iryna; Gladka, Zoya; Kotlyarov, Volodymyr; Teschl, Gerald. Long-time asymptotics for the Korteweg–de Vries equation with step-like initial data. *Nonlinearity* 26 (2013), no. 7, 1839–1864.
- [26] Fokas, A.; Fuchssteiner, B. Symplectic structures, their Backlund transformation and hereditary symmetries. *Physica D* 4 (1981), 47–66.
- [27] Germain P., Pusateri F., and Rousset F. Asymptotic stability of solitons for mKdV. [arXiv:1503.093143](https://arxiv.org/abs/1503.093143), 2015.
- [28] Grunert K., Holden H., Raynaud X. Global conservative solutions of the Camassa – Holm equation for initial data with nonvanishing asymptotics. [arXiv:1106.4125v1](https://arxiv.org/abs/1106.4125v1) [math.AP] 21.06.2011
- [29] Gurevich, A. V.; Pitaevskii, L. P. Decay of Initial Discontinuity in the Korteweg-de Vries Equation. (Russian) *JETP Letters* (1973), **17/5** 193
- [30] Khruslov E Ya 1976 Asymptotics of the solution of the Cauchy problem for the Korteweg de Vries equation with initial data of step type. *Matem. Sbornik (New Series)* **99(141):2** 261–281
- [31] Khruslov E Ya and Kotlyarov V P 1994 Soliton asymptotics of nondecreasing solutions of nonlinear completely integrable evolution equations *Spectral operator theory and related topics* Adv. Soviet Math. **19** Amer. Math. Soc. Providence, RI 129–180

- [32] Kotlyarov V., Minakov A. Riemann – Hilbert problem to the modified Korteweg – de Vries equation: Long-time dynamics of the step-like initial data. *Journal of Mathematical Physics* 51, 093506, 2010.
- [33] V. Kotlyarov and A. Minakov, Step-initial function to the mKdV equation: hyperelliptic long-time asymptotics of the solution, *Journal of mathematical physics, analysis, geometry*, 2012, 8/1, P. 38–62.
- [34] V. Kotlyarov and A. Minakov, Modulated elliptic wave and asymptotic solitons in a Cauchy problem to the modified Korteweg-de Vries equation, *Journal of Physics A: Mathematical and Theoretical*, **48**, 2015, 305201, 35 pp.
- [35] Minakov A. Riemann–Hilbert problem for the Camassa–Holm equation with step-like initial data, *J. Math. Anal. Appl.* (2015)
- [36] Mizumachi, Tetsu; Tzvetkov, Nikolay L²-stability of solitary waves for the KdV equation via Pego and Weinstein’s method. Harmonic analysis and nonlinear partial differential equations, 33–63, *RIMS Kôkyûroku Bessatsu*, B49, Res. Inst. Math. Sci. (RIMS), Kyoto, 2014
- [37] Pego, Robert L.; Weinstein, Michael I. Asymptotic stability of solitary waves. *Comm. Math. Phys.* 164 (1994), no. 2, 305–349.
- [38] A. B. Shabat and V. E. Zakharov, Integration of nonlinear equations of mathematical physics by inverse scattering problem method, I,II, *Funct. anal. and applications*, **8/3**, 1974, **13/3**, 1979. (in Russian).
- [39] Tzvetkov, Nikolay On the long time behavior of KdV type equations [after Martel-Merle]. *Séminaire Bourbaki*. Vol. 2003/2004. Astérisque No. 299 (2005), Exp. No. 933, viii, 219–248.